Introduction to Modern Physics

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Introduction to Modern Physics: Physics 311

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Part I

The Birth of Modern Physics

Chapter 1

What is Physics

Physics is the study of matter, energy and change. The physical world is studied through experimentation. It is through experimentation that all physical theories are tested and confirmed. Unlike math, its close cousin, Physics only makes sense in relation to what occurs in the possible in the physical world. A physical theory may be elegant or satisfying but it if does not conform to events then it is of no value.

One example can be found in electrostatics. Electrostatic fields are dependent upon the separation of positive and negative charges. Each charge exists as either a positive or negative monopole. If two charges are placed on either end of an imaginary rod, we can create what is known as a dipole. Magnetostatics would be more satisfying if there were individual magnetic "charges". Magnets only come in dipoles each south pole is accompanied by a north pole. If we could isolate either magnetic pole. Then magnetostatics would be formulated exactly the same as electrostatics. However, magnetic monopoles are not found in nature.

Mathematics provides a frame work and a language for physics. In applying math it needs to be remembered that physics is an experimental science. Often times math gives an avenue for theorists. As an example we are familiar with 3 physical dimensions (x,y,z) but we might imagine 4, or 5, or 10 or a million similar dimensions. One consequence of Einstein's special relativity is the addition of time as another dimension. Instead of space existing in 3 dimension (x,y,z), we have a four dimensional space-time system, (x,y,z,t).

1.1 Classical Physics and Modern Physics

Classical Physics is the name we give to the physical theories generated before the twentieth century. Classical physics is comprised primarily of two areas, mechanics and electro-magnetism. Understanding how physics progressed to the point where modern physics begins is important to understanding modern physics. Many of the concepts in modern physics are extensions of ideas that find their origin in classical physics.

Modern Physics is the name we give to the Theoretical advancements in physics since the beginning of the 20th Century, popularly known as Relativity and Quantum Mechanics. In this book, we will examine some of the seminal theoretical and experimental papers that formed the basis of modern physics.

Some concepts that will be developed include,

- The Special and General theories of Relativity
- The electron
- Quantization of energy
- Photo-electric effect
- The Bohr Atom and spectroscopy
- Compton Scattering
- Electron Free and Bound states
- Atoms in a Magnetic Field
- Heisenberg
- Schrdinger's Equation
- The Copenhagen interpretation of Quantum Mechanics
- Quantum Statistics
- Semiconductors
- The Standard Model

1.2 Assumptions of Science

In order to have a meaningful science we need to start with three assumptions. These assumptions are important to experimentation.

1. Common Perception
2. Uniformity of space and time
3. Natural Causes

The first assumption common perception, means that two independent observers observing the same experiment under the same conditions with agree on their observations. To facilitate agreements these observations must be quantified.

Uniformity of space means that physical laws are the same regardless of the position. It is possible that two experiments under different conditions may appear different, but we assume that the rules are the same and it is only the conditions of the experiment and the the observation that change the outcome.

The last assumption is strictly speaking a limitation on science that first two assumptions describe assumptions about the nature of the systems studied. The assumption of natural causes limits science to describing those phenomena that are the result of the nature of the universe. Supernatural events, those events that are outside of the universe cannot be studied since there are singular and therefor not repeatable.

Some scientists take this assumption to the point where there is assumed to be nothing outside of the universe. Since this cannot be shown to be true or false by experimentation. It is a theological and not a scientific statement.

Chapter 2

Classical Mechanics

2.1 Physics Before the 19th Century

One might say the golden era of science really began with the Galileo, who was ushered in the area of the experimental scientist. Before Galileo, science was largely philosophical, meaning theory was neither the result nor accompanied experiment. The result of this was a theories that were speculative and limited in their ability to explain physical phenomena. (for an excellent summary of the development of classical physics see "Introduction to Modern Physics by Richtmyer, et. al. 1969).

After Galileo, the giant who dominated physics for nearly 200 years (and is very important today) was Isaac Newton. Newton, among other things, was the first to unify the theory of motion and gravitation. Newton linked the acceleration a body feels on the earth (apocryphally an apple) to the motion of celestial bodies.

Newton proposed a simple set of principles;

1. **Law of Inertia: (also known as Galileo's Law of Inertia)** An object in constant motion stays in constant motion unless acted on by an outside unbalanced force.

2. The acceleration of an object is proportional to the net force applied to the object.

$$\vec{F} = m\vec{a} \tag{2.1}$$

   The proportionality constant is a quantity we will define as the mass, m.

3. For every force there is an equal and opposite force,

$$\vec{F}_{12} = -\vec{F}_{21} \tag{2.2}$$

4. Newton's law of gravitation can be written as,

$$F = -G\frac{m_1 m_2}{r^2}. \tag{2.3}$$

Which simply states that two bodies of masses, m_1 and m_2¡ exert a force on each other proportional to the product of their masses and the inversely proportional to the square of the distance between their mass centers.

Newton also did work in optics and mathematics.

2.2 Experiment and Observables

Our understanding of the physical world is predicated on those quantities which can be observed under the controlled environment of experimentation. We refer to these as quantities as observables.

All measurements are made relative to a system of coordinates. When an experiment is performed we make arbitrary choices about where the origin is placed and when a clock is started. This is known as the reference frame. A frame of reference (or reference frame) is a set of coordinates (x,y,z) that define the position of a body.

In the case of a train, we might define a reference frame that is stationary with respect to the train. We would then observer any object sitting on the train is stationary. However, we might define a reference frame that is stationary with respect to the platform. Now, the objects that are stationary with respect to the train are moving with respect to the platform. Similarly this effects bodies that are moving with respect to the train If a ball is dropped on the train to an observer on the train the ball appears to fall straight down, while an observer on the platform sees the ball travel in a parabolic path.

For the observer on the platform the motion is the vector sum of the motion of ball relative to the train and the train relative to the platform. Classically, the principle of relative motion was defined by Galileo,

- **Galilean Relativity**: When two observers are in constant relative motion ($v_{relative} = constant$) to each other they will observe the same physical laws.

The uniformity of space implies homogeneity, and that physical laws should remain the same under translation and rotation. In physics an objects position at a given instant is specified by a point of three coordinates $\vec{p}_1 = (x, y, z)$ The laws of mechanics then summarize as,

$$m\frac{d^2x}{dt^2} = F_x, m\frac{d^2y}{dt^2} = F_y, m\frac{d^2z}{dt^2} = F_z \tag{2.4}$$

If a series of phenomena are measured from two points the two observers should agree on the same laws. Each observer will measure the experiment from their own coordinate system.

We want to show that a transformation from one system to another will not change the laws of mechanics. For simplicity we will talk about two systems that have their axes aligned parallel, K (x,y,z) and K'(x', y', z'). To transform from K to K',

$$x' = x - a, y' = y, z' = z \tag{2.5}$$

In order for Newton's laws to remain unchanged undertranslation,

$$F_{x'} = F_x, F_{y'} = F_y, F_{z'} = F_z \tag{2.6}$$

By inspection we see that the last two remain the same. Looking at the transformation,

$$x' = x - a \tag{2.7}$$

Applying a variation in time,

$$\frac{dx'}{dt} = \frac{d(x-a)}{dt} = \frac{dx}{dt} - \frac{da}{dt}. \tag{2.8}$$

since a is a constant $da/dt = 0$ and,

$$\frac{dx'}{dt} = \frac{dx}{dt} \tag{2.9}$$

and correspondingly,

$$\frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2} \tag{2.10}$$

$$F_{x'} = F_x \tag{2.11}$$

We say that the laws of physics are symmetric under translation.

Lets examine rotation. The points in K' rotated in the x,y plane at an angle of θ relative to K.

$$x' = x\cos\theta + y\sin\theta \tag{2.12}$$

$$y' = y\cos\theta - x\sin\theta \tag{2.13}$$

$$z' = z. \tag{2.14}$$

simply the force F in K' is given by,

$$F_{x'} = F_x\cos\theta + F_y\sin\theta \tag{2.15}$$

$$F_{y'} = F_y\cos\theta - F_x\sin\theta \tag{2.16}$$

$$F_{z'} = F_z. \tag{2.17}$$

Now calculate d^2x/dt^2 when θ is a constant, then,

$$\frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt^2}\sin\theta \tag{2.18}$$

$$\frac{d^2y'}{dt^2} = \frac{d^2y}{dt^2}\cos\theta - \frac{d^2x}{dt^2}\sin\theta \tag{2.19}$$

Multplying by the mass and we clearly get,

$$F_{x'} = F_x \cos\theta + F_y \sin\theta \tag{2.20}$$
$$F_{y'} = F_y \cos\theta - F_x \sin\theta \tag{2.21}$$

Newton's laws are symmetric under rotation.

Under a Galilean transformation a body moving with a speed v in reference frame S, observed from reference frame S' moving with speed v' in relation to S, will appear to move at the speed,

$$u = v - v' \tag{2.22}$$

If S' is moving with speed $-v'$ (moving in opposite direction of S). The speed of the body will appear to be,

$$u = v + v' \tag{2.23}$$

Mathematically, this is a simple transformation from one coordinate system S to another S', a point in S' can be related to a point in S by the boost,

$$x' = x - v't \tag{2.24}$$

Now,

$$\frac{dx'}{dt} = \frac{dx}{dt} - v \tag{2.25}$$
$$\frac{d^2x'}{dt} = \frac{d^2x}{dt^2} \tag{2.26}$$

Newton's laws are symmetric under a galilean boost.

Homework 1

Show how Newton's law of inertia is violated for a reference frame where two bodies are in free-fall but at rest with each other.

From the homework problem it can be shown that Newton's Laws are consistent for boosts corresponding to constant velocities. In Fact, the definition of just such a reference frame borders on the philosophical. What is meant by a constant velocity, since we know that each measurement must be made relative to some reference frame. Newton defined an absolute rest frame in which his laws hold true, then for any other reference frame in constant relative motion to that absolute frame, the physical laws would true. (absolute motion is motion with respect to absolute space, a purely theoretical concept, while relative motion is motion relative to some chosen object) Similarly, Newton defined absolute time.

Now, all experiments are done in some relative reference frame since we cannot determine and/or do not know the "absolute rest frame". Since the law of inertia can be shown to be true by experiment, there must exist reference frames for which the laws of mechanics hold, in particular the law of inertia. These reference frames are known as inertial reference frames.

2.3 Inertial Reference Frames (IRF)

1. The law of inertia can be verified by experiment in an IRF.
2. Frame is not accelerating relative to other IRFs.
3. Any frame which is moving with uniform velocity relative to an IRF is an IRF

This leads the a set of transformations between a point (x, y, z) in one inertial frame S and a point (x', y', z') in another inertial frame S'. Where S' is moving with speed v' relative to S. (align the reference frames such that all axes are parallel to the respective primed axis and v' is on the x, x' direction.)

$$x' = x - v't \tag{2.27}$$
$$y' = y' \tag{2.28}$$
$$z' = z \tag{2.29}$$
$$t' = t \tag{2.30}$$
$$\tag{2.31}$$

To be complete we include the time in the transformation as we need to know the time a particle is at a particular position in order to fully understand its motion. This set of transformations is known as a Galilean transformation or a boost transformation.

The full set of any transformations must include translations (as listed above) and rotations. Newton's laws must be invariant under both. It can be shown that no mechanical experiment can detect any intrinsic difference between inertial reference frames.

2.3.1 Symmetries

So far we examined transformations related to translation, boosts and rotations. All 3 showed what we called symmetries. Symmetry is important to physics, in quantum mechanics a symmetry corresponds to a conservation law.

symmetry	conservation law
translation in space	momentum
translation in time	Energy
Rotation through a fixed angle	angular momentum
Uniform velocity in a straight line (Lorentz transformation)	

Chapter 3

Classical Electro-Magnetism

3.1 Maxwell and Electro-Magnetism

The next major unifying work was done by James Clerk Maxwell, who brought together different experimental and theoretical of electricity and magnetism. I don't want to slight the great work by physicists between Newton and Maxwell. Maxwell is notable however for combining the work of Ampere, Gauss and Faraday. into a set of equations that now bare Maxwell's name. Maxwell's equation's also showed that the electric and magnetic fields traveled in waves at the speed of light.

3.1.1 Maxwell's Equations in the integral form

$$\oint_s E_n dA = \frac{1}{\epsilon_0} Q_{inside} \tag{3.1}$$

$$\oint_s B_n dA = 0 \tag{3.2}$$

$$\oint_c E\dot{d}l = -\frac{d}{dt}\int_s B_n dA \tag{3.3}$$

$$\oint_c B\dot{d}l = \mu_0 I + \mu_0\epsilon_0 \frac{d}{dt}\int_s E_n dA \tag{3.4}$$

3.1.2 Maxwell's Equations in the derivative form

$$\nabla \cdot E = \frac{1}{\epsilon_o}\rho \tag{3.5}$$

$$\nabla \cdot B = 0 \tag{3.6}$$

$$\nabla \times E = = -\frac{\partial B}{\partial t} \tag{3.7}$$

$$\nabla \times B = \mu_o J + \mu_o \epsilon_o \frac{\partial E}{\partial t} \tag{3.8}$$

in regions where there is no charge or current, we write Maxwell's equations as,

$$\nabla \cdot E = 0 \tag{3.9}$$

$$\nabla \cdot B = 0 \tag{3.10}$$

$$\nabla \times E = = -\frac{\partial B}{\partial t} \tag{3.11}$$

$$\nabla \times B = \mu_o \epsilon_o \frac{\partial E}{\partial t} \tag{3.12}$$

Now we apply the operator $\nabla\times$ to the second two equations to get,

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E = \nabla \times (\frac{\partial B}{\partial t}) \tag{3.13}$$

$$= -\frac{\partial}{\partial t}(\nabla \times B) = -\mu_o \epsilon_o \frac{\partial^2 E}{\partial^2 t} \tag{3.14}$$

since $\nabla \cdot E = 0$;

$$\nabla^2 E = \mu_o \epsilon_o \frac{\partial^2 E}{\partial^2 t} \tag{3.15}$$

and similarly,

$$\nabla \times (\nabla \times B) = \nabla(\nabla \cdot B) - \nabla^2 B = \nabla \times (\frac{\partial E}{\partial t}) \tag{3.16}$$

$$= -\frac{\partial}{\partial t}(\nabla \times E) = -\mu_o \epsilon_o \frac{\partial^2 B}{\partial^2 t} \tag{3.17}$$

since $\nabla \cdot B = 0$;

$$\nabla^2 B = \mu_o \epsilon_o \frac{\partial^2 B}{\partial^2 t} \tag{3.18}$$

Which is the same as the wave equation,

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{3.19}$$

the solution of which is,

$$f = f_o \sin(kx - \omega t) \tag{3.20}$$

Where the velocity v,

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}, \tag{3.21}$$

is same as the speed of light $c = 3.00 \times 10^8 m/s$, and

$$k = \frac{2pi}{\lambda} \tag{3.22}$$

$$\omega = 2\pi f \tag{3.23}$$

The discovery that maxwell's constant c for electromagnetic waves was the same as the speed of light in a vaccum, lead to the conclusion that light was an electromagnetic wave of the same form as x-rays, and radiowaves, just on different wavelenths.

3.2 Reference Frames?

One difficulty between Newtonian Mechanics and Maxwell's equations are the complex relationship between Electric and magnetic fields. Imagine a magnet in motion and a conductor at rest, this results in a electric field with an associated energy and a current in the conductor. While a simple change of reference gives a stationary magnet and moving conductor, which leads to no Electric field (and an abscence of associated energy) + an EMF in the conductor and a current.

We note that Maxwell's equations are independent of direction and propogation and contain no reference to the frame in which the motion happens, In fact experimentation shows that reguardless of the frame in which Maxwell's equations are measured. The results are the same, and the speed of light is measured at the same value. How can this be as the theory of Inertial reference frames (Newtonian Mechanics) implies that the value of the speed of light should vary by the speed of the relative frames, it doesn't.

- What does the fact that Maxwell's equations holds for all inertial frames imply?
- What does this mean to the concepts of space and time?

The implication is that the speed of light is the same for all inertial frames. We have to ask now is there a transformation for which Newton's laws hold, and the speed of light is constant for all inertial frames?

3.3 Implications of the Speed of Light

1. As yet unknown medium for light to travel through
2. There is no medium and light travels with a constant speed requarless of reference frame. Requires a new relativity.

3.4 Vector Calculus

Gradiant,

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} \tag{3.24}$$

Divergence,

$$\nabla \cdot \vec{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \tag{3.25}$$

Curl

$$\nabla \times \vec{f} = (\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z})\hat{\imath} + (\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x})\hat{\jmath} + (\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y})\hat{k} \tag{3.26}$$

Laplacian.

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{3.27}$$

Homework 2
Show that Maxwell's equations do not hold under a boost transformation.

3.5 EM and Optics

Part II

Relativity

Chapter 4

Special Relativity

Read: On the Electrodynamics of Moving Bodies, Does the Inertia of a Body depend upon its Energy-content?

It has been shown, that the equations governing electrodynamics do not transform via Galilean transformations. We ask the question is there a transformation for which the laws of mechanics and electrodynamics hold true? This question was first answered by Lorentz, who constructed a mathematical transformation, that satisfied the requirement of maintaining the structure of electrodynamics. There was however no physical support for such a transformation.

In 1905, Albert Einstein provided the physical support for the Lortenz transformation and in his seminal paper, "On the Electrodynamics of Moving Bodies" (1905), which introduced the concept of special relativity.

In the discussion of Einstein's paper we will use for clarity his notion with differs from the standard notation in use for a transformation between one frame K given by the coordinates (x, y, z, t) and another frame K' given by (ξ, η, ζ, τ).

4.1 Two Postulates

Einstein postulated that physical theory should be mediated by two conditions.

1. (Principle of Relativity) Laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of the two systems of coordinates in uniform transitory motion.

2. Any ray of light moves in the "stationary" system of coordinates with the determined velocity c, whether the ray be emitted by a stationary source or a moving one.

The first postulate states that any physical law should be stated in such a way that definition of a reference frame is not necessary for its validity. (In actuality the first postulate is limited to coordinate systems that are moving with constant relative motion. Einstein left it to general relativity to include accelerating frames.)

The second postulate states a physical law as the constraint, namely that the anything moving with the speed c in one frame will be measured as moving at the speed c in all reference frames. This constraint is necessary to ensure that Maxwell's equations are acceptable in the this new formalism.

Since physics is an experimental science, theories must be experimentally verified, so we take a moment to discuss measurements. To measure the length of an object I hold a rule to it and measure the positions of the edges of the object relative to the rule. The measurement is made by judging the position of the rule and the position of the object simultaneously.

4.1.1 Simultaneity

All judgments of time are judgments of simultaneous events. "A train arriving at 7:00 really means the pointing of the small hand of my watch to 7 and the arrival of the train are simultaneous."

It is easy to see that as events become more distant it is difficult to evaluate times. Because the speed of light is finite, we cannot synchronize with a distant event. Thus at point A we can measure events near A and at point B we can measure events near B.

In a frame, where a clocks A and B are stationary with respect to each other the clocks are synchronized if the time required for light to travel from A to B $(t_B - t_A)$ is equal to the time required to travel from B to A. $(t'_A - t_B)$,

$$t_B - t_A = t'_A - t_B. \tag{4.1}$$

1. if A synchronizes with B, B synchronizes with A.
2. If A synchronizes with B and C, B synchronizes with C.

Thus the speed of light is given by,

$$c = \frac{2AB}{t'_A - t_A} \tag{4.2}$$

Why do we spend time on the concept of measurement and simultenaity? The reason is simple ....

4.1.2 Measurement of Length

How does our view simultaneity effect the measurement of length.

1. An observer moving with a measuring rod and the rod which is to be measured. conclusion: The rod measures l, the same as if it is at rest. (This is simply an expression of the principle of relativity.)

2. A rod moving relative to an observer and a measuring rod. Conclusion: the length of the rod is not measured as l. Because of the finite speed of light the ends of the rod are not measured simultaneously, therefore the rod will appear smaller.

4.1.3 Simultaneity in moving frames

[INSERT GRAPHICAL REPRESENTATION]

Only observers at rest will agree w/ one another on the simultaneity of two events. The condition of simultaneity becomes,

$$\frac{1}{2}(t'_A + t_A) = t_B \tag{4.3}$$

Imagine a light pulse leaving at $t_A = 5$ and returning at $t'_A = 10$, in the frame where the clocks are at rest with each other the light pulse reflects at a time halfway between t'_A and t_A or 7.5, From the condition of simultaneity we find,

$$t_B = \frac{1}{2}(t'_A + t_A) \tag{4.4}$$

$$t_B = \frac{1}{2}(10 + 5) = 7.5 \tag{4.5}$$

$$\tag{4.6}$$

Here WE NEED TO INCLUDE A DISCUSSION OF TIME

4.2 Building a suitable transformation: Taken from Relativity by Einstein

Again we take 2 reference frames K and K' that are in K' is moving with velocity v relative to K.

If at time $t = \tau = 0$, a pulse of light is sent out along the $x-$axis, from $x = \xi = y = \eta = x = \zeta = 0$, then the position of light pulse may be described by,

$$x = ct \tag{4.7}$$

which can be written as,

$$x - ct = 0 \tag{4.8}$$

Similarly in the K',

$$\xi = c\tau \tag{4.9}$$

$$\xi - c\tau = 0 \tag{4.10}$$

Since this one event must simultaneously satisfy both equations (4.8) and (4.10), they must satisfy,

$$\xi - c\tau = \lambda(x - ct) \tag{4.11}$$

similarly for a pulse moving in the $-x$ direction, we obtain the relation,

$$\xi + c\tau = \lambda(x + ct). \tag{4.12}$$

Using equations (4.11) and (4.12) solve for ξ and τ,

$$\xi = \frac{\lambda + \mu}{2}x - \frac{\lambda - \mu}{2}ct, \tag{4.13}$$

$$c\tau = \frac{\lambda + \mu}{2}ct - \frac{\lambda - \mu}{2}x. \tag{4.14}$$

since λ and μ are arbitrary functions, we will substitute,

$$\gamma = \frac{\lambda + \mu}{2}, \tag{4.15}$$

$$\Gamma = \frac{\lambda - \mu}{2}. \tag{4.16}$$

Allowing us to write ξ and $c\tau$ as,

$$\xi = \gamma x - \Gamma ct, \tag{4.17}$$

$$c\tau = \gamma ct - \Gamma x. \tag{4.18}$$

Looking at the origin of K' where $\xi = 0$ gives,

$$x = \frac{\Gamma c}{\gamma}t \tag{4.19}$$

Since this point must have a position of,

$$x = vt. \tag{4.20}$$

This then gives the velocity of K' relative to K,

$$v = \frac{\Gamma c}{\gamma} \tag{4.21}$$

Now rewrite equations (4.17, 4.18),

$$\xi = \gamma x - \gamma vt, \tag{4.22}$$

$$c\tau = \gamma x - \gamma\frac{v}{c}t. \tag{4.23}$$

In order to determine the function a, imagine a frame K' that is at moving with velocity v relative to frame K, the transforms of x and t are now given by,

$$\xi = \gamma(x - vt), \tag{4.24}$$

$$\tau = \gamma(t - \frac{v}{c^2}x). \tag{4.25}$$

Further, there is a frame $K''(x', y', z', t')$ that is moving relative to K' with a velocity of $-v$, then the transform of ξ and τ into K'' is given by,

$$x' = \gamma(\xi + v\tau), \tag{4.26}$$

$$t' = \gamma(\tau + \frac{v}{c^2}\xi). \tag{4.27}$$

Then transform from K to K'' is given by,

$$x' = \gamma(\gamma(x - vt) + \gamma v(t - \frac{v}{c^2}x), \tag{4.28}$$

$$t' = \gamma(\gamma(t - \frac{v}{c^2}x) + \gamma\frac{v}{c^2}(x - vt)). \tag{4.29}$$

Simplifying,

$$x' = \gamma^2(x - vt + vt - \frac{v^2}{c^2}x), \tag{4.30}$$

$$t' = \gamma^2(t - \frac{v}{c^2}x + \frac{v}{c^2}x - \frac{v^2}{c^2}t), \tag{4.31}$$

and,

$$x' = \gamma^2 x(1 - \frac{v^2}{c^2}), \tag{4.32}$$

$$t' = \gamma^2 t(1 - \frac{v^2}{c^2}), \tag{4.33}$$

Since K and K'' must be at rest with respect to each other, $x = x'$, and $t = t'$, a must satisfy,

$$1 = \gamma^2(1 - \frac{v^2}{c^2}), \tag{4.34}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{4.35}$$

Allowing us to rewrite our transforms as,

$$\xi = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{4.36}$$

$$\tau = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{4.37}$$

for simplicity we introduce $\beta = \frac{v}{c}$,

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{4.38}$$
$$\xi = \gamma(x - \beta ct), \tag{4.39}$$
$$\tau = \gamma(t - \frac{\beta}{c}x). \tag{4.40}$$

Since the motion is orthogonal to y and z they cannot be affected by the relative velocity so the transforms above are supplemented by,

$$\eta = y \tag{4.41}$$
$$\zeta = z \tag{4.42}$$

We can see that a necessary condition of the second postulate of relativity (constancy of the speed of light) is that a light pulse in 3 dimensions forms a spherical wave front, if it propagates from the origin we write,

$$r = \sqrt{x^2 + y^2 + z^2} = ct \tag{4.43}$$
$$x^2 + y^2 + z^2 - c^2t^2 = 0 \tag{4.44}$$

The same light wave in K',

$$r' = \sqrt{\xi^2 + \eta^2 + \zeta^2} = c\tau \tag{4.45}$$
$$\xi^2 + \eta^2 + \zeta^2 - c^2\tau^2 = 0 \tag{4.46}$$

They must satisfy the relation,

$$\xi^2 + \eta^2 + \zeta^2 - c^2\tau^2 = \sigma(x^2 + y^2 + z^2 - c^2t^2) \tag{4.47}$$

Homework 3
Show the derived transformation holds for $\sigma = 1$.

4.3 Length Contraction

Let us consider how the transformation of two points leads to a change in length, and a change in time. Consider a rod of length $L_p = \xi_2 - \xi_1$ at rest in frame K'.

$$\xi_1 = \gamma(x_1 - vt), \tag{4.48}$$
$$\xi_2 = \gamma(x_2 - vt), \tag{4.49}$$
$$\xi_2 - \xi_1 = \gamma(x_2 - x_1 - (vt - vt)), \tag{4.50}$$
$$\xi_2 - \xi_1 = \gamma(x_2 - x_1), \tag{4.51}$$
$$L_p = \gamma L. \tag{4.52}$$

Where L_p is the length of the rod as measured in its rest frame, and L is the length as measured in frame where the rod is moving. The length of a moving rod is shorter than its rest length.

4.3.1 Example: Length Contraction

If a rod is 1m at rest and moving at a speed 0.8c, what is the new length of the rod?
given: $L_p = 1m \quad v = 0.8c \quad v/c = 0.8 \quad \gamma = 1/\sqrt{1-\frac{v^2}{c^2}}$

$$L_p = \gamma L \tag{4.53}$$

$$L = \frac{L_p}{\gamma} = \sqrt{1-\frac{v^2}{c^2}}L_p \tag{4.54}$$

$$= (1m)\sqrt{1-(0.8)^2} = 0.6m \tag{4.55}$$

4.4 Time Dilation

Now examining the difference between 2 times, consider two times τ_1 and τ_2 happening at the same place ξ' in K',

$$t_1 = \gamma(\tau_1 + \frac{v}{c^2}\xi'), \tag{4.56}$$

$$t_2 = \gamma(\tau_2 + \frac{v}{c^2}\xi'), \tag{4.57}$$

$$\Delta t = t_2 - t_1 = \gamma(\tau_2 - \tau_1) + \gamma(\frac{v}{c^2}\xi' - \frac{v}{c^2}\xi'), \tag{4.58}$$

$$\Delta t = t_2 - t_1 = \gamma\Delta\tau = \gamma\Delta t_p. \tag{4.59}$$

Because the events happening at τ_1 and τ_2 occur at the same place, they are stationary, this time difference is referred to as the proper time, t_p. The time Δt as measured from a frame where the there is movement between the two events the time measured is slower than the proper time.

4.4.1 Example: Time Dilation

If a stationary event takes $1.00s$ in K' and K' is traveling at $v = 0.8c$ in relation to frame K, find the time the event takes in frame K.
given: $\Delta t_p = 1s \quad v = 0.8c \quad \beta = /c = 0.8 \quad \beta = 1/\sqrt{1-\beta^2}$

$$\Delta t = \gamma\Delta t_p = \frac{1}{\sqrt{1-\beta^2}}t_p \tag{4.60}$$

$$= \frac{1}{\sqrt{1-(0.8)^2}}(1.0s) = 1.7s \tag{4.61}$$

4.4.2 Graphical derivation of Time Dilation

The time it takes like to strike a mirror as distance d away and return is,

$$\Delta t = \frac{2d}{c} \tag{4.62}$$

If the source and mirror are now placed in a ship moving perpendicular to the direction of the light at a speed of v, The time the light takes will appear as,

$$\Delta t' = \frac{2D}{c} \tag{4.63}$$

where,

$$D = \sqrt{d^2 + (\frac{v\Delta t'}{2})^2} \tag{4.64}$$

combine this with Eq. (4.63) gives,

$$\begin{aligned}
(\frac{c\Delta t'}{2})^2 &= d^2 + (\frac{v\Delta t'}{2})^2 && (4.65)\\
&= (\frac{c\Delta t}{2})^2 + (\frac{v\Delta t'}{2})^2 && (4.66)\\
(\frac{c\Delta t'}{2})^2 - (\frac{v\Delta t'}{2})^2 &= (\frac{c\Delta t}{2})^2 && (4.67)\\
(\frac{c^2}{2})\Delta t'^2(1 - \frac{v^2}{c^2}) &= (\frac{c^2}{2})\Delta t^2 && (4.68)\\
\Delta t'^2(1 - \frac{v^2}{c^2}) &= \Delta t^2 && (4.69)\\
\Delta t'^2 &= \gamma \Delta t^2 && (4.70)
\end{aligned}$$

4.4.3 Proper length and proper time

The proper length of a rod is the length of a rod measured in its rest frame with a ruler that is at rest with respect to the rod.

The proper time is the the interval measured between events happening at the same point.

Homework 4
Item Imagine a ship and a rod observed to be moving together with velocity v. Does the rod measure the proper length?

Homework 5
A ship is observed by moving with velocity v, is measured with a rod stationary with respect to the observer, does the rod measure the proper length?

Homework 6
A ship moving at a speed of $v = 0.80c$ relative to a particular satelite, If the satalite measures a time of $2.00s$ for the ship to pass the satellite what is the proper time?

Homework 7
A ship moving at a speed of $v = 0.80c$ travels from Earth to a star the clock on the ship measures a time of 90 years. What is the proper time?

Homework 8
How fast must a muon travel such that it's mean lifetime is $46\mu s$, if its rest lifetime is $2\mu s$.

Homework 9
A spaceship travels to a star 95 light years away at a speed of $2.2 \times 10^8 m/s$. How long does it take to get there (a) as measured on the earth? (b) as measured by a passenger on the spaceship?

4.4.4 Time discrepancy of an event synchronized in K as viewed in K' separated by $x_2 - x_1$

The time discrepancy of an event synchronized in K viewed in K' where the events are separated by $(x_2 - x_1)$.

$$\tau_1 = \gamma(t_1 - \frac{Vx_1}{c^2}) \tag{4.71}$$

$$\tau_1 = \gamma(t_2 - \frac{Vx_2}{c^2}) \tag{4.72}$$

$$\tau_1 - \tau_1 = \gamma(t_2 - t_1) - \beta(\frac{Vx_2}{c^2} - \frac{Vx_1}{c^2}) \tag{4.73}$$

$$t_2 - t_1 = \frac{V}{c^2}(x_2 - x_1) \tag{4.74}$$

$$\Delta t = \frac{V}{c^2}L_p \tag{4.75}$$

4.4.5 Time discrepancy of events separated by Δt and Δx in K as viewed by K'.

4.5 Doppler Effect

A Source stationary in K emits Nwaves in a time Δt. If K moves with speed v with respect to frame K', while in frame K' a receiver sits stationary with respect to K'.

What is the shift in frequency received in K' from that emitted in K. The wavelength received is calculated as,

$$\lambda' = \frac{c\Delta t_R - v\Delta t_R}{N}, \tag{4.76}$$

Where Δt_R is the time for the wave to travel from the source to the receiver in the receiver's rest frame. Now we calculate the received frequency as,

$$\begin{aligned} f_R &= \frac{c}{\lambda'} = \frac{c}{c-v}\frac{N}{\Delta t_R} && (4.77)\\ &= \frac{1}{1-\frac{v}{c}}\frac{N}{\Delta t_R}, && (4.78)\end{aligned}$$

Given $N = f_S \Delta t_S$, and $\Delta t_S = \Delta t_p$,

$$\begin{aligned} f_R &= \frac{1}{1-\frac{v}{c}}\frac{f_S\Delta t_S}{\Delta t_R}, && (4.79)\\ &= \frac{1}{\gamma}(\frac{1}{1-\frac{v}{c}})f_S, && (4.80)\\ &= \frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}}f_S && (4.81)\\ f_R &= \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}f_S && (4.82)\end{aligned}$$

For a source approaching a receiver,

$$f_R = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}f_S \tag{4.83}$$

As functions of wavelength,

$$\begin{aligned} f &= \frac{c}{\lambda} && (4.84)\\ \frac{f_R}{f_S}_{approaching} = \frac{\lambda_S}{\lambda_R} &= \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} && (4.85)\\ \frac{f_R}{f_S}_{receding} = \frac{\lambda_S}{\lambda_R} &= \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} && (4.86)\\ & && (4.87)\end{aligned}$$

4.5.1 Example: Doppler Effect

Assuming a distant galaxy is moving at $v = 0.64c$ and it gives light at a wavelength of $\lambda_0 = 656nm$, What is the new wavelength if the galaxy is approaching? or receding? Approaching:

$$\frac{\lambda_S}{\lambda_R} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \tag{4.88}$$

$$= 307nm \tag{4.89}$$

Which is a shift toward the color blue.
Receding:

$$\frac{\lambda_S}{\lambda_R} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \tag{4.90}$$

$$= 1399nm \tag{4.91}$$

which is a shift toward red.

If most spectra of distance galaxies appear to be longer in wavelength than expected the universe is expanding.

4.6 Twin Paradox

Assume pair of twins, Abraham and Bill, Bill goes on a trip at a speed of v to a distant star, after reaching the star Bill turns around and returns at a speed of $-v$. What are their relative ages?

Assume the star is 4 lightyears away, and bill travels at a speed $v = 0.80c$. In Abraham's time frame, bill travels 5 years before turning around.

$$t = \frac{L_p}{v} = \frac{4}{0.80} = 5years \tag{4.92}$$

While bill's clock is slower,

$$\Delta\tau = \frac{\Delta t}{\gamma} = \frac{5y}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{5}{5/3} = 3years \tag{4.93}$$

the return trip takes the same amount of time in Abraham's frame. So 10years has passed for Abraham, while on 6 years has passed for Bill.

DRAW A SPACE TIME DIAGRAM

Chapter 5

Velocity, momentum and Energy

Having found the point transformations,

$$x = \gamma(\xi + v\tau) \tag{5.1}$$
$$t = \gamma(\tau + \frac{v}{c^2}\xi) \tag{5.2}$$
$$\xi = \gamma(x - vt) \tag{5.3}$$
$$\tau = \gamma(t - \frac{v}{c^2}x) \tag{5.4}$$

5.1 Velocity Transformations

How does one transform a velocity of $\vec{u}$ in frame K to a velocity of u'_x in frame K', if K' is which is moving at v with respect to K. Clearly the velocities are given by,

$$u'_x = \frac{\Delta\xi}{\Delta\tau} \quad , \quad u'_y = \frac{\Delta\eta}{\Delta\tau} \tag{5.5}$$
$$u_x = \frac{\Delta x}{\Delta t} \quad , \quad u_y = \frac{\Delta y}{\Delta t} \tag{5.6}$$

Then,

$$\Delta\xi = \xi_2 - \xi_1 = \gamma(x_2 - vt_2) - \gamma(x_1 - vt_1) \tag{5.7}$$
$$\Delta\xi = \gamma(\Delta x - v\Delta t) \tag{5.8}$$

and,

$$\Delta\tau = \gamma(\Delta t - \frac{v}{c^2}\Delta x) \tag{5.9}$$

Substituting (5.8) and (5.9) into equation (5.5) gives,

$$
\begin{aligned}
u'_x &= \frac{\gamma(\Delta x - v\Delta t)}{\gamma(\Delta t - \frac{v}{c^2}\Delta x)} && (5.10)\\
&= \frac{1}{\Delta t}\Big(\frac{\Delta x - v\Delta t}{1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}}\Big) && (5.11)\\
&= \frac{u_x - v}{1 - \frac{v}{c^2}u_x} && (5.12)
\end{aligned}
$$

Substituting (5.8) and (5.9) into equation (5.5) gives,

$$
\begin{aligned}
u'_y &= \frac{\Delta y}{\gamma(\Delta t - \frac{v}{c^2}\Delta x)} && (5.13)\\
&= \frac{1}{\gamma}\frac{u_y}{1 - \frac{v}{c^2}u_x} && (5.14)\\
&= \frac{u_y\sqrt{1-\beta^2}}{1 - \frac{v}{c^2}u_x} && (5.15)
\end{aligned}
$$

Then the velocity transforms are,

$$
\begin{aligned}
u'_x = \frac{u_x - v}{1 - \frac{v}{c^2}u_x} \quad &, \quad u'_y = \frac{u_y\sqrt{1-\beta^2}}{1 - \frac{v}{c^2}u_x} && (5.16)\\
u_x = \frac{u'_x - v}{1 + \frac{v}{c^2}u'_x} \quad &, \quad u_y = \frac{u'_y\sqrt{1-\beta^2}}{1 + \frac{v}{c^2}u'_x} && (5.17)
\end{aligned}
$$

5.1.1 Example: transforming the velocity of an object moving at the speed of light

Now, let us apply this to an object moving at c in frame S'.

$$
\begin{aligned}
u'_x &= c && (5.18)\\
u_x &= \frac{u'_x + v}{1 + \frac{v}{c^2}u'_x}, && (5.19)\\
&= \frac{c + v}{1 + \frac{v}{c^2}c}, && (5.20)\\
&= \frac{c + v}{\frac{1}{c}(c + v)} = c && (5.21)
\end{aligned}
$$

Homework 10
A distant galaxy is moving away from us at a speed of $1.85 \times 10^7 m/s$. Calculate the fractional redshift ($\frac{\lambda' - \lambda_0}{\lambda_0}$) in the light from this galaxy.

Homework 11
Two spaceships are approaching each other (a) if the speed of each is 0.6c relative to the earth, what is the speed relative to each other? (b) if the speed of each relative to the earth is 30,000m/s, what is the speed of one relative to the other?

5.1.2 Transformation of the angle of a light ray

Assume a light ray is traveling in frame K at and angle θ. How is this ray viewed from K' which is moving at a speed of v relative to K.

$$u_x = c\cos\theta \tag{5.22}$$
$$u_y = c\sin\theta \tag{5.23}$$

Applying the velocity transforms we find the x' and y' coordinates of the light ray,

$$u_x = \frac{c\cos\theta + v}{1 + \frac{v}{c}\cos\theta} \tag{5.24}$$
$$u_y = \frac{c\sin\theta\sqrt{1-\beta^2}}{1 + \frac{v}{c}\cos\theta} \tag{5.25}$$

The angle θ' can then be calculated from either the sin or tan functions,

$$\tan\theta' = \frac{u'_y}{u'_x} = \left(\frac{c\sin\theta\sqrt{1-\beta^2}}{1 + \frac{v}{c}\cos\theta}\right)\left(\frac{1 + \frac{v}{c}\cos\theta}{c\cos\theta + v}\right) \tag{5.26}$$
$$= \frac{\sin\theta\sqrt{1-\beta^2}}{\cos\theta + \beta} \tag{5.27}$$
$$\sin\theta' = \frac{\sin\theta\sqrt{1-\beta^2}}{1 + \beta\cos\theta} \tag{5.28}$$

5.2 Transformation of the Energy of Light Rays

We can define the energy of a light complex per unit volume as,

$$\frac{A^2}{8\pi} \tag{5.29}$$

If the volume of the light complex were the same in two reference frames, S and S' then the ratio of energy in the S' frame to the S frame would be,

$$\frac{A'^2}{A^2} \tag{5.30}$$

However, the volume is not the same, because of time and length contraction, What is a sphere in the S frame,

$$(x - lct)^2 + (y - mct)^2 + (z - nct)^2 = R^2, \tag{5.31}$$

where l, m, n are direction of the wave normal cosines. This surface is an ellipsoid in S',

$$\beta\xi - l\xi\frac{v}{c^2})^2 + (\eta - m\beta\xi\frac{v}{c^2})^2 + (\zeta - n\beta\xi\frac{v}{c^2})^2 = R^2 \tag{5.32}$$

The ratio of the volume of the light complex in S' to that in S is,

$$\frac{V_{S'}}{V_S} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c}\cos\phi} \tag{5.33}$$

Now we can write the ratio of the energy in the primed frame to that in the unprimed frame as,

$$\frac{E'}{E} = \frac{A'^2 S'}{A^2 S} = \frac{1 - \frac{v}{c}\cos\phi}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5.34}$$

And if $\phi = 0$ then,

$$\frac{E'}{E} = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \tag{5.35}$$

Which is similar to the ratio of the frequencies, thus we observe, that

$$E \propto f \tag{5.36}$$

5.3 Does the Inertia of a Body depend upon its Energy-content?

We found previously that the ratio energy of a plane wave of light with energy E in the rest frame and E' in a frame moving with speed v with respect to the rest frame. frame is,

$$\frac{E'}{E} = \frac{1 - \frac{v}{c}\cos\phi}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5.37}$$

Now, the source gives off plane waves with energy $\frac{1}{2}L$ at an angle ϕ to the direction of v, if the source has a energy E_0 before emission and E_1 after emission of two plane waves in opposite directions (so that there is no change in momentum), we can write the conservation of energy as,

$$E_0 = E_1 + \frac{1}{2}L + \frac{1}{2}L \tag{5.38}$$

Similarly in the moving frame we can write the energy as,

$$H_0 = H_1 + \frac{1}{2}L\frac{1-\frac{v}{c}\cos\phi}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{1}{2}L\frac{1-\frac{v}{c}\cos\phi}{\sqrt{1-\frac{v^2}{c^2}}} \tag{5.39}$$

$$= H_1 + \frac{L}{\sqrt{1-\frac{v^2}{c^2}}} \tag{5.40}$$

H and E can differ only by the kinetic energy and a constant, so,

$$H_0 - E_0 = K_0 + C \tag{5.41}$$

$$H_1 - E_1 = K_1 + C \tag{5.42}$$

So,

$$K_0 - K_1 = H_0 - E_0 - (H_1 - E_1), \tag{5.43}$$

$$= (H_0 - H_1) - (E_0 - E_1) \tag{5.44}$$

$$= \frac{L}{\sqrt{1-\frac{v^2}{c^2}}} - L \tag{5.45}$$

$$= L(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1) \tag{5.46}$$

Doing an expansion of the term $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$,

$$\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 1 - \frac{1}{2}\frac{v^2}{c^2} + \text{higher order terms} \tag{5.47}$$

neglecting the higher order terms we get,

$$K_0 - K_1 = \frac{1}{2}\frac{L}{c^2}v^2, \tag{5.48}$$

Therefore if a body gives off energy L in the from of radiation its mass diminishes by $\frac{L}{c^2}$,

$$\Delta m = \frac{L}{c^2}, \tag{5.49}$$

$$L = \Delta m c^2. \tag{5.50}$$

5.4 Momentum

If a constant force is applied to a body, under Newtonian mechanics we would expect the body to continue accelerating up to an infinite velocity. Special relativity, however, means that c is a limiting velocity. This means the inertia (mass) of a body must increase.

$$v \to c \tag{5.51}$$
$$m \to \infty \tag{5.52}$$

Then the mass of a body must be a function of its speed and mass at rest.

$$m_u = m_0 f(u), \tag{5.53}$$
$$m_u = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \tag{5.54}$$

the momentum can now be written as,

$$\vec{p} = m_u \vec{u} \tag{5.55}$$
$$= \gamma m_0 \vec{u}, \tag{5.56}$$
$$= \frac{m_0 u}{\sqrt{1-\frac{u^2}{c^2}}} \tag{5.57}$$

Here we take a moment to note that the momentum is dependent upon the velocity and not directly the relative speed between the frames.

1. $\vec{p}$ is conserved

2. as $\frac{u}{c} \to 0$, $\nrightarrow mu$

Now we will use p to calculate the kinetic energy,

$$K = \int_{u=0}^{u} \sum F ds, \tag{5.58}$$
$$= \int_{0}^{u} \frac{dp}{dt} ds, \tag{5.59}$$
$$= \int_{o}^{u} u dp, \tag{5.60}$$
$$= \int_{o}^{u} u d(\frac{m_0 u}{\sqrt{1-\frac{u^2}{c^2}}}), \tag{5.61}$$
$$d\Big(\frac{m_0 u}{\sqrt{1-\frac{u^2}{c^2}}}\Big) = m_0 \Big(1-\frac{u^2}{c^2}\Big)^{-3/2} du, \tag{5.62}$$
$$K = \int_{o}^{u} u m_0 \Big(1-\frac{u^2}{c^2}\Big)^{-3/2} du, \tag{5.63}$$
$$= m_0 c^2 \Big(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1\Big), \tag{5.64}$$
$$K = \frac{m_0 c^2}{\sqrt{1-\frac{u^2}{c^2}}} - m_0 c^2 \tag{5.65}$$

Define,

$$E_0 = m_0 c^2 \tag{5.66}$$

We can write the energy as,

$$E = K + m_0 c^2, \tag{5.67}$$
$$= K + E_0, \tag{5.68}$$
$$= \frac{m_0 c^2}{\sqrt{1-\frac{u^2}{c^2}}} \tag{5.69}$$

The work done to move a mass from rest to the final energy,

$$\frac{m_0 c^2}{\sqrt{1-\frac{u^2}{c^2}}} = m_r c^2 \tag{5.70}$$

Which allows us to write a relativistic mass as,

$$m_r = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \tag{5.71}$$

Chapter 6

Four vectors and Lorentz invariance

6.1 Momentum

To preserve the concept of momentum conservation it must be shown that if momentum is conserved in one frame it is conserved in all inertial frames. We start with some observations about momentum.

1. Newtonian formulation is incorrect.
2. The relativistic form must reduce to the Newtonian from when $u << c$.
3. $\vec{p}$ is a vector in the same direction as $\vec{u}$.
4. $\Delta\vec{p} = 0$ for a collision or explosion for any inertial frame.

The last statement is a postulate of invariance. The quantity Δp is invariant under transformation. Since the Newtonian form is not invariant, we need a new formulation. What quantities have we studied are invariant under transformation? It was shown previously that the quantity describing the envelope of a light wave is invariant,

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2. \tag{6.1}$$

If a vector is written,

$$x_\mu = (x, y, z, ict) \tag{6.2}$$

Then,

$$x_\mu \cdot x_\mu = x^2 + y^2 + z^2 - c^2t^2 \tag{6.3}$$

And can be understood at the length, $L = x_\mu \cdot x_\mu$, x_μ or the radius of a four dimensional hypersphere. We can see that similar to Newtonian physics the length of x_μ is invariant.

Once our set of coordinate are written as a four vector, we can summarize the Lorentz transformation as a matrix operation.

$$\begin{bmatrix} x' \\ y' \\ z' \\ ict' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ ict \end{bmatrix} \tag{6.4}$$

Or written as,

$$x_\mu = \Lambda^\mu_\nu x_\nu \tag{6.5}$$

where,

$$\Lambda = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \tag{6.6}$$

Other invariants may be formed from,

- the sum, product, quotient, or difference of two Lorentz scalars
- 0
- the Dot product, $\vec{A}_\mu \cdot \vec{B}_\mu$ of any two four vectors.

6.1.1 Invariant interval

Suppose an event 1 occurs at (x_1, y_1, z_1, ict_1) and event 2 occurs at (x_2, y_2, z_2, ict_2). The difference,

$$\Delta x^\mu = x_2^\mu - x_1^\mu \tag{6.7}$$

is the displacement four-vector. And the invariant product of Δx^μ is called the invariant interval,

$$I = \Delta x_\mu \Delta x^\mu = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\Delta t^2 \tag{6.8}$$
$$= -c^2 t^2 + d^2 \tag{6.9}$$

Where t is the time interval between two events and d their spacial separation.

1. $I < 0$ the interval is time-like (events that occur at the same place but separated by time)
2. $I > 0$ the interval is space-like (events that occur at the same time but are separated by space.
3. $I = 0$ light-like events are connected by a signal traveling at the speed of light.

If $I < 0$ then there exists a system in which the events occur at the same point.

If $I > 0$ then there exists a system in which the events occur are synchronized. The boost velocity v would have to be greater than c to place them at the same place.

Since the Lorentz transformations include the time as well as space, we may treat time as another dimension of space. 3-Dimensions of normal space and one dimension of time, this is known as Minkowski-space.

Let us consider a light ray produced at the origin and moving to the right with a slope of c.

INSERT GRAPH

Consider a light pulse expanding in all directions on a plane, after sometime t. The tip of the light pulse can be expressed as $r = ct$. Similar to the above graph if we plot this ever expanding circle as a function of t as well, this ever expanding circle forms a cone around the time axis.

We might imagine, though we can't really draw it, A series of rays expanding in 3-D. Now, we have an expanding sphere in time with the time axis plotted in a 4-D hyper-cone.

Typically we plot ct rather than t, so that all dimensions have the same units.

We can also extend the time axis in to the past getting a past light cone which represents a shrinking sphere of light (non-physical). This gives us a double cone defined by $r^2 = c^2t^2$. This Minkowski space represents the past, present and future, the light cone separates the world in regions.

Minkowski space permits an easy way of representing all motions of a body. It does this by a single line know as a world line. This world line represents the potion of a body at any time. Since speeds of the body must be equal to or less than the speed of light. The angle the world line makes with the time axis will be less than the angle of the light cone.

The tangent of the world line gives its velocity.

The light cone represents all points in the future which can be reached from the current point, and all points in the past the could have reached the tip.

Since simultaneity depends upon the reference frame. Two events A_1 and B_1 are simultaneous relative to frame R, they are not simultaneous in R'. Recall previously the diagram in which we illustrated the condition of simultaneity. The line A_1B_1 is parallel to the $x-$axis but that same line as seen from a moving frame. $A_1'B_1'$ is not parallel to the $x'-$axis tilted lines indicate simultaneity in a moving frame, in Minkowski space-time that tilted line becomes a tilted $x - y - z - t$ space.

The now space for a moving observer in R' is tilted with respect to the now space of the observer at rest in R.

INSERT IMAGE

Therefore every event outside the light cone is simultaneous with the origin if seen from a suitable frame. The slope of the line connecting two events tells you if the invariant interval is time like (slope $> 1/c$) or space-like (slope$< 1/c$).

The minus sign in the invariant interval gives rise to an important geometric result. Minkowski space is hyperbolic, and in 4 dimensions a hyperboloid. Under rotation around a spatial axis, point p describes a circle in the other two spacial coordinates, given by $r^2 = x^2 + y^2$. Under a Lorentz rotation the interval $I = d^2 - c^2t^2$ is preserved. The locus of all point with a given I is a hyperboloid. Time like I have a hyperboloid made of two sheets, while a space-like hyperboloid consists of 1 sheet.

No, amount of transformation will carry a point from the lower sheet to the upper sheet of the time-like hyperboloid. This means that if the invariant interval between two events is time-like, the ordering is absolute; if the interval is space-like the ordering depends on the inertial system. The invariant interval between causally related events is always time-like.

Invariant interval and the proper-time

The invariant interval was defined as,

$$I = \Delta x_\mu^2 = \Delta x_\mu \Delta x^\mu = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\Delta t^2 \tag{6.10}$$

$$= -c^2\Delta t^2 + d^2 \tag{6.11}$$

Where a clock in our rest frame measures a movement of $\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$ in the time interval Δt between two events. By a clock in the objects rest frame the two events are separated spatially by $L' = x_2' - x_1' = 0$ and a time $\Delta t'$. Since this interval is measured by a clock stationary with respect to the points $x_2' = x_1'$, $\Delta t'$ measures the proper time.

Then we write,

$$c^2\Delta t'^2 = c^2\Delta t_p^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \tag{6.12}$$

$$\Delta t_p^2 = \Delta t^2\Big(1 - \frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{c^2\Delta t^2}\Big) \tag{6.13}$$

$$\tag{6.14}$$

Since,

$$\frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{\Delta t^2} = v \tag{6.15}$$

Leads to,

$$\Delta t_p^2 = \Delta t^2\Big(1 - \frac{v^2}{c^2}\Big) \tag{6.16}$$

$$\Delta t_p = \Delta t\sqrt{1 - \beta^2} \tag{6.17}$$

$$\frac{\Delta t_p}{\gamma} = \Delta t \tag{6.18}$$

This shows that the invariant interval is simply the square of the proper time,

$$I = -c\Delta t_p^2 \tag{6.19}$$

6.1.2 four-velocity

Previously we defined the velocity relative to quantities in the same frame,

$$u = \frac{dx}{dt}. \tag{6.20}$$

Let us define the four-velocity v_μ as,

$$\eta_\mu = \frac{\Delta x_\mu}{\Delta t_p} \tag{6.21}$$

where Δt_p is the proper time. The proper velocity is now measured relative to the time the traveler measures. Since,

$$\Delta t_p = \frac{\Delta t}{\gamma} \tag{6.22}$$

We find that the four-velocity is given by,

$$\eta_\mu = \gamma\frac{\Delta x}{\Delta t} = \frac{\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}, \tag{6.23}$$

$$\eta_4 = \frac{c}{\sqrt{1-\frac{u^2}{c^2}}} \tag{6.24}$$

In frames moving with velocity V, the four-velocity transform with,

$$\eta'_x = \gamma(\eta_x - \beta\eta_4) \tag{6.25}$$

$$\eta'_y = \eta_y \tag{6.26}$$

$$\eta'_z = \eta_z \tag{6.27}$$

$$\eta'_4 = \gamma(\eta_4 - \beta\eta_x) \tag{6.28}$$

or

$$\begin{bmatrix} \eta'_x \\ \eta'_y \\ \eta'_z \\ i\eta'_4 \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} \eta_x \\ \eta_y \\ \eta_z \\ i\eta_4 \end{bmatrix} \tag{6.29}$$

Transforming again with the rotation, Λ, in fact all four-vectors will transform using the matrix Λ.

6.1.3 Four-momentum

We will define the four-momentum as,

$$p_\mu = m_o eta_\mu \tag{6.30}$$

If we expand γ by the binomial expansion,

$$(1-x)^n = 1 - nx - \frac{n(n-1)}{2}x^2 + \cdots , \tag{6.31}$$

where $u << c$ then,

$$\begin{aligned} p &= \frac{mu}{\sqrt{1-\frac{u^2}{c^2}}} = mu(1-\frac{u^2}{c^2})^{-1/2}, & (6.32) \\ p &= mu(1-\frac{1}{2}(\frac{u}{c})^2 + \frac{3}{8}(\frac{u}{c})^4 + \cdots). & (6.33) \end{aligned}$$

As $\frac{u}{c} \to 0$ then, $p = mu$. The component of p_4 expands as,

$$\begin{aligned} p_4 &= \frac{mc}{\sqrt{1-\frac{u^2}{c^2}}} & (6.34) \\ &= mc + \frac{1}{2}m\frac{u^2}{c} + \cdots & (6.35) \end{aligned}$$

So that as $\frac{u}{c} \to 0$ then, the p_4c reduces to the kinetic energy $\frac{1}{2}mu^2$. Thus we write the kinetic energy as,

$$K = mc^2(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1) \tag{6.36}$$

We observe;

1. $\vec{p}$ parallel to $\vec{u}$
2. $\vec{p}$ goes to the classical result as $u << c$.

6.1.4 Conservation of Momentum

6.1.5 Classical momentum conservation under a Galilean transformation

Given the classical momentum of $p = mv$ examine a collision between two masses A and B, that result in two new mass C and D, then,

$$m_A v_A + m_B v_B = m_C v_C + m_D v_D. \tag{6.37}$$

Now, calculate the transformation to K' where the velocity transforms in the Galilean method by $v' = v - u$,

$$\begin{aligned} m_A v_A + m_B v_B &= m_C v_C + m_D v_D & (6.38) \\ m_A(v'_A - u) + m_B(v'_B - u) &= m_C(v'_C - u) + m_D(v'_D - u) & (6.39) \end{aligned}$$

Collecting like terms,

$$m_A v'_A + m_B v'_B - m_C v'_C - m_D v'_D + u(m_A + m_B - m_C - m_D) = 0 \tag{6.40}$$

The first term gives momentum conservation if the second term $m_A + m_B - m_C - m_D$ is zero. Classically this is known as mass conservation.

6.1.6 Relativistic Momentum Conservation

Let us examine a collision between two masses A and B, that result in two new mass C and D in K' with velocities $a', b', c'd'$ respectively, the momentum perpendicular to the boost is given by,

$$m_A \eta'_{A_y} + m_B \eta'_{B_y} = m_C \eta'_{C_y} + m_C \eta'_{D_y} \tag{6.41}$$

Since the transform of a 4-velocity perpendicular to the boost is given by, $\eta_y = \eta'_y$, then,

$$m_A \eta_{A_y} + m_B \eta_{B_y} = m_C \eta_{C_y} + m_C \eta_{D_y} \tag{6.42}$$

Therefore momentum is conserved in the y, z directions if the boost is in the x direction.

In the direction parallel to the boost we have transforms,

$$\eta'_1 = \gamma(\eta_1 - \beta\eta_4) \tag{6.43}$$
$$\eta'_4 = \gamma(\eta_4 - \beta\eta_1) \tag{6.44}$$

The momentum in the parallel direction is given by (dropping the indice 1),

$$m_A \eta'_A + m_B \eta'_B = m_C \eta'_C + m_C \eta'_D \tag{6.45}$$
$$m_A(\gamma(\eta_A - \beta\eta_{A_4})) + m_B(\gamma(\eta_B - \beta\eta_{B_4})) = \tag{6.46}$$
$$m_C(\gamma(\eta_C - \beta\eta_{C_4})) + m_D(\gamma(\eta_D - \beta\eta_{D_4})) \tag{6.47}$$

Collecting like terms we arrive at,

$$m_A\eta_A + m_B\eta_B - m_C\eta_C - m_D\eta_C + \beta(m_A\eta_{A_4} + m_B\eta_{B_4} - m_C\eta_{c_4} + m_D\eta_{D_4}) = 0 \tag{6.48}$$

The first term like in the classical solution is simply the conservation of momentum if, the second term is zero,

$$m_A\eta_{A_4} + m_B\eta_{B_4} - m_C\eta_{c_4} + m_D\eta_{D_4} \tag{6.49}$$

We observe,

$$\sum_i^N \frac{m_i c}{\sqrt{1 - \frac{u^2}{c^2}}} = 0 \tag{6.50}$$

is conserved. Multiplied by c gives,

$$\sum_{i}^{N} \frac{m_i c^2}{\sqrt{1-\frac{u^2}{c^2}}} = 0 \tag{6.51}$$

In equation (6.36) the relationship to kinetic energy was shown giving equation (6.51) the meaning of conservation of energy, We now define the energy as,

$$E = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} \tag{6.52}$$

and the mass as,

$$\mathcal{M} = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}. \tag{6.53}$$

The quantity,

$$E = \mathcal{M}c^2 \tag{6.54}$$

is conserved. m_0 is sometimes called the rest energy and $E_0 = m_0 c^2$ as the rest energy.

From the relations,

$$\vec{p} = \frac{mu}{\sqrt{1-\frac{u^2}{c^2}}} \tag{6.55}$$

$$E = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} \tag{6.56}$$

It can be shown,

$$pc = uE. \tag{6.57}$$

And the four-momentum can be written as,

$$p_\mu = (p_1, p_2, p_3, i\frac{E}{c}) \tag{6.58}$$

Which transforms using Λ,

$$\begin{bmatrix} p_1' \\ p_2' \\ p_3' \\ i\frac{E'}{c} \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ i\frac{E}{c} \end{bmatrix} \tag{6.59}$$

or

$$\begin{gathered} p_1'c = \gamma p_1 c - \beta\gamma E \\ p_2'c = p_2'c \\ p_3'c = p_3'c \\ E' = -\beta\gamma p_1 c + \gamma E \end{gathered} \tag{6.60}$$

Since p_μ is a four-vector, we construct an invariant for the momentum

$$\begin{aligned}
p^\mu p_\mu &= \vec{p}\cdot\vec{p} - (\frac{E}{c})^2 \\
\vec{p}\cdot\vec{p} - (\frac{E}{c})^2 &= \frac{m_0^2 \vec{u}\cdot\vec{u}}{1-\frac{u^2}{c^2}} - \frac{m_0^2 c^2}{1-\frac{u^2}{c^2}} \\
&= \frac{m_0^2 c^2}{1-\frac{u^2}{c^2}}(\frac{u^2}{c^2} - 1) \\
&= -m_0^2 c^2 \\
\vec{p}\cdot\vec{p} - (\frac{E}{c})^2 &= -m_0^2 c^2 \qquad (6.61)
\end{aligned}$$

or as it is more commonly written,

$$E^2 - p^2c^2 = m_0^2 c^4 \qquad (6.62)$$

6.1.7 Particles of zero mass

Given,

$$\begin{aligned}
\vec{p}c &= \frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}} \qquad (6.63) \\
\vec{E} &= \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} \qquad (6.64) \\
pc &= uE \qquad (6.65) \\
E^2 - p^2c^2 &= m_0^2 c^4 \qquad (6.66)
\end{aligned}$$

We can see that as $\frac{u}{c} \to 1$, the mass has to vanish, leading to to relationship between energy and momentum for a mass-less particle as,

$$E = pc \qquad (6.67)$$

Homework 12
How much rest mass must be converted to energy to (a) produce 1J (b) to keep a 100-W light bulb burning for 10 years.

Homework 13
(2.67 in Tipler) The sun radiates energy at the rate of 4×10^{26} W. Assume that this energy is produced by a reaction whose net result is the fusion of 4 H nuclei to form 1 He nucleus, with the release of 25 MeV for each He nucleus formed. Calculate the sun's loss of rest mass per day.

Homework 14
(2.70 in Tipler) Show that

$$d\Big(\frac{m_o u}{\sqrt{1-u^2/c^2}}\Big) = m_0\Big(1-u^2/c^2\Big)^{-3/2} du \tag{6.68}$$

Homework 15
(2.79 in Tipler) For the special case of a particle moving with speed u along the y axis in the frame S, show that its momentum and energy in frame S' are related to its momentum and energy in S by the transformation equations

$$\begin{aligned} p'_x &= \beta\Big(p_x - \frac{VE}{c^2}\Big) && (6.69)\\ p'_y &= p_y && (6.70)\\ p'_z &= p_z && (6.71)\\ \frac{E'}{c} &= \beta\Big(\frac{E}{c} - \frac{Vp_x}{c}\Big) && (6.72) \end{aligned}$$

6.2 Mechanical Laws

We can see that this formulation of the momentum poses a difficulty for Newton's laws of mechanics. Let us examine Newton's second law which we will write as,

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{6.73}$$

In terms of the force the work is given by,

$$W = \int \vec{F}\cdot d\vec{l} \tag{6.74}$$

Now combining we get,

$$\begin{aligned} W &= \int \frac{d\vec{p}}{dt}\cdot d\vec{l} = \int \frac{d\vec{p}}{dt}\cdot\frac{d\vec{l}}{dt}dt = \int \frac{d\vec{p}}{dt}\cdot\vec{u}dt \\ \frac{d\vec{p}}{dt}\cdot\vec{u}dt &= \frac{d}{dt}\Big(\frac{mu}{\sqrt{1-\frac{u^2}{c^2}}}\Big)u = \frac{m\vec{u}}{(1-\frac{u^2}{c^2})^{3/2}}\cdot\frac{d\vec{u}}{dt} = \frac{d}{dt}\Big(\frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}\Big) = \frac{dE}{dt} \\ W &= \int \frac{dE}{dt}dt = E_{final} - E_{initial} && (6.75) \end{aligned}$$

6.2.1 Motion under a constant force

Imagine a particle of mass m that is subjected to a constant force $\vec{F}$. In classical mechanics we imagine that this implies a constant acceleration leading to an infinite velocity. Since this is a clear violation of the second principle of special relativity, we reexamine this problem in terms of the relativistic momentum. Then, solving for x, as a function of time.

$$\frac{dp}{dt} = F \tag{6.76}$$

implies,

$$Ft + constant \tag{6.77}$$

If the particle starts from rest then at time $t = 0$, $p = 0$ and the constant is zero.

$$\vec{p} = \frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}} = Ft \tag{6.78}$$

solving for u, we arrive at,

$$u = \frac{(F/m)t}{\sqrt{1 - (\frac{Ft}{mc})^2}} \tag{6.79}$$

integrate again,

$$\begin{aligned} x &= \frac{F}{m}\int_0^t \frac{t'}{\sqrt{1-(\frac{Ft'}{mc})^2}}dt' = \frac{mc^2}{F}\sqrt{1-(\frac{Ft'}{mc})^2}\ |_0^t \\ &= \frac{mc^2}{F}\left[\sqrt{1-(\frac{Ft'}{mc})^2} - 1\right] \end{aligned} \tag{6.80}$$

Classically the solution is a parabola $x = (F/2m)t^2$ this solution is a hyperbola. As a constant force is applied the velocity of the particle asymptotically approaches c.

6.2.2 Cyclotron Motion

The trajectory of a charged particle in a uniform magnetic field is circular or cyclotron motion. The magnetic force is given by,

$$F = quB \tag{6.81}$$

In special relativity we need to calculate the centripetal force.

$$F = \frac{dp}{dt} = p\frac{d\theta}{dt} = p\frac{u}{R} \tag{6.82}$$

now,

$$
\begin{aligned}
quB &= p\frac{u}{R} && (6.83)\\
p &= qBR && (6.84)
\end{aligned}
$$

Homework 16
Show that for uniform circular motion $dp = pd\theta$

6.3 Proper Force

It would be useful to produce a force that is a four-vector, we take as the proper force the variation of the, of the momentum with respect to the proper time. You should recall that is this a similiar definition to the proper velocity,

$$K^\mu = \frac{\Delta p^\mu}{\Delta t_p} \tag{6.85}$$

Then,

$$\vec{K} = \gamma \vec{F} \tag{6.86}$$

And the 4th component is,

$$K^4 = \frac{\Delta p^4}{\Delta t_p} = \frac{1}{c}\frac{dE}{dt} \tag{6.87}$$

Chapter 7

General Relativity

Einstein cites 2 reasons for a new theory of gravitation,

1. Special Relativity Applies to all forces but gravity, Gravity is not invarient under special relativity

2. To construct an inertial reference frame (IRF), one must eliminate all forces, but gravity cannot be eliminated

Falling can eliminate the force of gravity, if all other forces are also eliminated the falling person becomes an inertial observer. However, now the world appears to be in an accelerated frame. We can check the law of inertia in free fall (in a small lab over a short distance) the objects appear at rest or in constant relative motion and we seem to be moving in a straight line, in agreement with the law of inertia.

If the law of inertia is true in free fall, and in systems under the acceleration of gravity, we conclude the law of inertia hold relative to an accelerated frame.

7.1 Absolute Acceleration?

It is then natural to ask if acceleration is absolute or relative?

- Newton argued for absolute acceleration (2 buckets)

- Ernst Mach Argued that if the universe had only these 2 pails they would be no way to distinquish between the pails and symmetry would demand the same sufrace on the water of both pails.

Einstein surmized that acceration should be relative just like velocity and position.

7.2 2 principles of General Relativity

- Principle of Covarience

- Principle of Equivalence

All Physical laws must be expressed in a form covarient with respect to an arbitrary coordinate transformation.

7.2.1 Principle of Covarience

Under special relativity inertial reference frames are preferable due to the simplicity of the laws of nature as expressed in inertial frames.

- We would like a relativity that is general in any frame.
- We expect the laws of nature to be the same in and out of an inertial frame.

This generaliztion destroys the uniqueness of the concept of space and time measurements.

7.2.2 Results of a transforming rotation in S.R.

$K_O(X, Y, Z, T)$ and K rotating at ω with respect to K_0. Upon transformation of $K_0 \rightarrow K$ Z remains unchanged, in K_0,

$$X = R\cos\theta, \tag{7.1}$$
$$Y = R\sin\theta, \tag{7.2}$$

while in K we write,

$$x = r\cos(\theta - \omega T) \tag{7.3}$$
$$y = r\sin(\theta - \omega T) \tag{7.4}$$
$$\tag{7.5}$$

$r = R$ because it is orthognal to ω.

The length of an arc transforms as,

$$Rd\theta \rightarrow \frac{rd\theta}{\sqrt{1 - \frac{(r\omega)^2}{c^2}}} \tag{7.6}$$

The measure of the line element connecting (r, θ) to $(r + dr, \theta + d\theta)$ in K is written as,

$$dl^2 = dr^2 + \frac{r^2 d\theta}{1 - \frac{(r\omega)^2}{c^2}} \tag{7.7}$$

If $dr = 0$ and $d\theta = 0$ the ratio of the circumference of a circle to its radius becomes,

$$\frac{circumference}{r} = \frac{2\pi}{\sqrt{1 - \frac{(r\omega)^2}{c^2}}} > 2\pi \tag{7.8}$$

We recognize that a measuring rod is $l = l/\beta$ in length in K and so it measures a larger circumference by a factor of β.

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There are several conclutions we can draw;

- For an observer in K space is not homogeneous or isotropic.
- The measure of time is effected.
- The transfer from an inertial fram to a non-inertial frame results in a complete loss of objectivity in spacial and temporal measurements.

Thus we would prefer to apply relativity across all transformations.

7.2.3 Principle of Equivalence

No dynamical experiment can distinquish between the so-called ficticious (inertial) force arising in system K_1 and the so-called real (gravitational) force prevailing in system K_2.

Gravity is detected when a reference frame is not freely falling. We experience gravity only because the earth prevents us from falling toward its center, gravity is a consequence of the choice of reference frame.

This leads to the concept that the inertial mass of a body is equal to its gravitational mass.

Eovtos measured the force of gravity on a body suspended, and the force of centerfuge arrising from the rotation of the earth. He found the inertial mass and the gravitational mass to be the same to 5 parts in 10^9. Further experiments have refined this to 1 part in 10^{11}.

The conclusion is that inertial mass is of gravitational origin, and leads to the bending of light by gravity.

If we cannot tell the difference between a frame under the acceration of gravity and one being accelerated by a rocket. We examine the path of a light ray in such a frame. In a rocket if it is accelerated, a light ray emmited perpendicular to the acceleration, will appear on the wall slightly lower than height at which it was emitted, because of the movement of the frame. If we cannot tell the difference, then in a frame (stationary) under the acceleration of gravity, we would expect the light to strike the wall slightly lower than emitted, meaning that the light ray bent slightly.

The first observation of such a bending of light rays occured in 1919 (5 years after the publication of General relativity). Observervations of a star on the edge of an eclipse and the same star when the sun was far away from the star. This position of the star showed a shift. Shifts as predicted by gravitational theories,

- Newton's Law predicts no bending of the light

- Newton's Law + special relativity - predicts bending exactly 1/2 the amount predicted by general relativity.
- GR predicts bending (for the particular star $1.75''$)

The results were about 2.0 to 1.6 arc seconds with and uncertainty of $\pm 0.3''$. The observation is consistant only with the theory of GR. More-recently we have other gravity lens observations.

Remember that in SR the velocity of light is a constant, meaning constant speed and direction. If a light ray is bent how can this be?

How is a straight line defined?

If we define a straight line as the shortest distance between 2 points, then picture the surface of a sphere, the shortest distance then follows a greatest circle. In general this is known as a geodesic.

This light follows the geodesic on the surface of space-time.

We now picture the sun not as a sphere but as a depression in space.

7.3 Experimental support for General Relativity

- Gravitational Red Shift - increased gravitational pull slows a clock
- Comparison of Clock at Boulder moves faster than the one at the Royal Greenwich Observatory
- Precession of the orbits of planets-
  - If we calculate the orbits of the planets using the theory of general relativity, we find a precession in the orbits of the perihelion that is not present in the calculations based upon Newtonian gravitation.
  - Observed on Mercury (1845) 43.0 arc sec per century $\approx 1\%$
  - Venus 8.63 arc sec per century
  - earth 3.8 arc sec per century

7.3.1 Black Holes

Bodies emit gravitational radiation - a vibration in the curvature of space-time propogation with the speed of light

Is there evidence of such radiation? Bianary Stars, pulsars

Black Hole a region in space with a gravitational pull larger than a critical value,

$$v_{escape} > c \tag{7.9}$$

All radiation is absorbed, and none is emitted.

In Newtonian mechanics we would calculate the escape velocity as,

$$v_e = \sqrt{\frac{2GM}{R}} \tag{7.10}$$

Then if we set the escape speed equal to the speed of light and solve for the radius, we obtain the critical radius R_S, called the Schwarzchild radius,

$$R_S = \frac{2GM}{c^2}. \tag{7.11}$$

For an object of the mass of our sun its radius would have to be about 3km.

Part III

Origins of Quantum Mechanics

Chapter 8

The Quantum Hypothesis

We have seen how the rules of classical physics break down when objects move at speeds comparable to the speed of light. The laws of classical physics similarly break down when they are applied to microscopic systems (Atomic sized).

This has lead to the Theory of Quantum Mechanics, which is the only extant theory that adequately describes such phenomena.

Quantum mechanics was primarily hashed out between (1881-1932). The origin of quantum mechanics are in thermodynamics.

Some Important Dates

- 1895 Röngen discovers x-rays
- 1896 Becquerl discovers nuclear radiation
- 1897 J.J. Thompson discovers the electron and measures e/m, shows the electron is part of an atom
- 1900 Plank explains blackbody radiation through energy quantization and a new constant h
- 1905 Einstein explains the photoelectric effect
- 1907 Einstein applies energy quantization to temperature dependence of heat capacities
- 1908 Rydberg and Ritz generalize Balmer's formulas
- 1909 Millikin's oil drop experiment
- 1911 Rutherford's discovery of the nucleus
- 1913 Bohr Model of Hydrogen atom
- 1914 Mosely Analyzes x-ray spectra using Bohr model

- 1914 Frank and Hertz Demonstrate atomic energy quantization
- 1916 Millikin verifies Einstein's photoelectric equations
- 1923 Compton explains x-ray scattering
- 1924 de Broglie proposes electron waves of $\lambda = h/p$
- 1925 Shrödinger develops mathematics of electron wave mechanics
- 1925 Heisenberg invents matrix mechanics
- 1925 Pauli exclusion principle
- 1927 Heisenburg uncertainty principle
- 1928 Gamow and Condon and Gurney Apply Quantum Mechanics to explain alpha-decay lifetimes
- 1928 Dirac proposes relativistic quantum mechanics and predicts the positron
- 1932 Chadwick discovers the neutron
- 1932 Anderson discovers the positron

We know from experience that an object that is heated gives off radiation. This is known as thermal radiation. Usually this radiation is in the IR range.

- $1000K \sim$ Red hot
- $2000K \sim$ Yellowish-white

The actual light is a collection of the wavelength's given off, a continuous spectrum with a dominate wavelength.

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We can simulate this with the use of a Blackbody, as we recall a blackbody is an ideal absorber/emitter of radiation. We can think of a black body a cavity with a small hole, radiation that falls on the hole enters the cavity, but does not leave.

As the temperature increases the dominate wavelength sifts, this is known as Wein's displacement law,

$$\lambda_{max} T = 2.90 \times 10^{-3} mK \tag{8.1}$$

Thermal radiation results from the oscillations of atoms near the surface, classically these oscillations are not restricted and may take any value. The classical calculations predict an intensity proportional to,

$$I \propto \frac{1}{\lambda^4} \tag{8.2}$$

This theory is good at large wavelengths, but we can see that as, $\lambda \to 0$ $I \to \infty$ Then an infinite amount of energy is being radiated at any temperature. It also deviates from the experimental value. This is known as the ultraviolet catastrophe

Max Plank (German 1858-1947) formulated an idea bases on the mathematical conjecture that these oscillators are limited to certain energies given by,

$$E_n = nhf \ n = 1, 2, 3, \cdots \tag{8.3}$$

where f is the frequency of the oscillator. The oscillator then emits energy when it changes from one state to another. Since the energy comes in discrete amounts this is called a quantum of energy. The quantity h is a constant known as Plank's constant.

$$h = 6.63 \times 10^{-34} Js \tag{8.4}$$

The results of Plank's theory matches the experimental values, and solves the issue of the UV-catastrophe. Plank was not satisfied though, believing that his mathematical trick, did not have physical significance. History would prove Plank wrong on the physical significance and Plank won the Nobel Prize in 1918.

8.1 The energy density

The total energy density of a system of oscillators may be calculated from the density of standing waves in a cavity, $G(\nu)d\nu$, over the interval ν to $\nu + d\nu$, and the average energy per mode, $\bar{U}$,

$$u(\nu)d\nu = G(\nu)\bar{U}d\nu \tag{8.5}$$

The density of standing waves in a cavity may be calculated as,

$$G(\nu)d\nu = \frac{8\pi^2\nu^2}{c^3}d\nu \tag{8.6}$$

The classical result may be derived from the equipartition theorem, where the average energy for each degree of freedom for a gas at temperature T is $1/2k_BT$. A one dimensional harmonic oscillator has 2 degrees of freedom, one corresponding to the kinetic energy and one to the potential energy. Thus, the classical average for a standing wave is,

$$\bar{U} = k_BT \tag{8.7}$$

and we can write the energy per unit volume as,

$$u_\nu d\nu = G(\nu)\bar{k}_BTd\nu = \frac{8\pi^2\nu^2k_BT}{c^3}d\nu \tag{8.8}$$

This is known as they Rayleigh-Jeans Formula and may be written in terms of wavelength as,

$$u_\lambda d\lambda = \frac{8\pi^2k_BT}{\lambda^4}d\lambda \tag{8.9}$$

the energy per unit volume is given by,

$$U = \int_0^\infty u_\lambda d\lambda \tag{8.10}$$

$$= \int_0^\infty \frac{8\pi^2 k_B T}{\lambda^4} d\lambda \tag{8.11}$$

Clearly $U \to \infty$ as $\lambda \to 0$

8.1.1 number density of modes in a cavity

On a one dimensional string of length L, the wave equation, and its solution, may be written as,

$$\frac{\partial^2}{\partial x^2} f = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \tag{8.12}$$

$$f = A \sin \omega t \cos kx \tag{8.13}$$

In order to satisfy the boundary conditions of $f(0) = 0$ and $f(L) = 0$,

$$k = \frac{n\pi}{L} x \tag{8.14}$$

where,

$$n = \frac{2L}{\lambda}, \text{ where } n = 0, 1, 2, 3, \cdots \tag{8.15}$$

If we generalize this result to a three dimensional cavity of volume $V = L^3$, where, the standing waves are electro-magnetic modes, then,

$$E_x = E_{0_x} \sin \omega t \cos(\frac{n_x \pi}{L} x) \sin(\frac{n_z \pi}{L} z) \sin(\frac{n_z \pi}{L} z) \tag{8.16}$$

$$E_x = E_{0_y} \sin \omega t \sin(\frac{n_x \pi}{L} x) \cos(\frac{n_z \pi}{L} z) \sin(\frac{n_z \pi}{L} z) \tag{8.17}$$

$$E_x = E_{0_z} \sin \omega t \cos(\frac{n_x \pi}{L} x) \sin(\frac{n_z \pi}{L} z) \cos(\frac{n_z \pi}{L} z) \tag{8.18}$$

Then $\vec{n} = (n_x, n_y, n_z)$, and

$$n^2 = n_x^2 + n_y^2 + n_z^2 = (\frac{2L}{\lambda})^2 \tag{8.19}$$

To find the number of wavelengths that excede the given value of λ_m,

$$n_x^2 + n_y^2 + n_z^2 \leq \frac{4L^2}{\lambda_m^2}, \tag{8.20}$$

The point (n_x, n_y, n_z) is on the surface of a sphere of radius $\frac{2L}{\lambda}$, to calculate the number points contained in a volume of radius $\frac{2L}{\lambda_m}$ we simply integrate over the area of a sphere for with a radius between n and $n + dn$,

$$N = \int_0^{\frac{2L}{\lambda}} 4\pi n'^2 dn' \tag{8.21}$$

$$N = \frac{4}{3}\pi\left(\frac{8L^3}{\lambda_m^3}\right) \tag{8.22}$$

We are only concerned with the density of states in the positive octant, dividing by 8, and allowing for two transverse polarizations per wave, we obtain,

$$n = \frac{8\pi}{3}\frac{1}{\lambda_m^3} \tag{8.23}$$

where n is the volumetric number density. And correspondingly the number of points between λ and $\lambda + \Delta\lambda$,

$$dn = \frac{8\pi}{\lambda^4} d\lambda \tag{8.24}$$

Writing in terms of the frequency, $\lambda = c/\nu$ and $d\lambda = c/\nu^2 d\nu$, The number density of modes in a cavity is,

$$G(\lambda)d\lambda = \frac{8\pi}{\lambda^4} d\lambda \tag{8.25}$$

$$G(\nu)d\nu = 8\pi\frac{\nu^2}{c^3} d\nu \tag{8.26}$$

8.1.2 Solution by Micro-canonical Ensemble

Assume that the energy of a quantum mechanical oscillator is,

$$E = s\epsilon = sh\nu, \text{ where } s = 0, 1, 2, 3 \ldots \tag{8.27}$$

This means than an oscillator emits radiation at a frequency of ν when it drops from one state to the next lower one. Each energy bundle of $h\nu$ is known as a quantum of energy. And the total energy may be written in terms of the average energy of each oscillator, U,

$$U_N = NU \tag{8.28}$$

U_N is a discrete quantity of an integral number of finite equal parts, each such part having an energy element ϵ. The black-body is made of N oscillators (with energy $\epsilon = h\nu$, with P energy elements distributed among the oscillators.

$$U_N = P\epsilon \tag{8.29}$$

And by extension,

$$\frac{P}{N} = \frac{U}{\epsilon} \tag{8.30}$$

Label the resonators $n = 1, 2, 3, \cdots, N$, and place P energy elements in the N oscillators. For example, one such complex for $N = 10$ and $P = 100$ would be,

1	2	3	4	5	6	7	8	9	10
4	28	0	11	9	22	4	5	5	12

The number Γ of all possible permutations (A perumation is different if the same numbers are in different locations) is given by,

$$\Gamma = \frac{N(N+1)(N+2)\cdots(N+P-1)}{1\cdot 2\cdot 3\cdots P} = \frac{(N+P-1)!}{N!P!} \tag{8.31}$$

The probability, W that a system of N resonators has energy U_N is proportional to Γ the possible permutations of all distributions of P among N resonators. The entropy of a system is given by,

$$S_N = k \log W + \text{ const.} \tag{8.32}$$

Which can be related to average entropy of one oscillator, S, as,

$$S_N = NS \tag{8.33}$$

And writing in terms of Γ,

$$S_N \quad = \quad k \log \Gamma = k \log\left(\frac{(N+P-1)!}{N!P!}\right) \tag{8.34}$$

Applying Stirling's Approximation to the highest order,

$$N! = N^N \tag{8.35}$$

Γ can be written as,

$$\Gamma = \frac{(N+P)^{N+P}}{N^N P^P}. \tag{8.36}$$

And the entropy is written as,

$$S_N = k\{(N+P)\log(N+P) - N\log N - P\log P\} \tag{8.37}$$

Writing S_N in terms of P/N we get,

$$S_N = kN\{(1+\frac{P}{N})\log(1+\frac{P}{N}) - \frac{P}{N}\log PN\} \tag{8.38}$$

We have from equations (8.33) and (8.30),

$$S = k\{(1+\frac{U}{\epsilon})\log(1+\frac{U}{\epsilon}) - \frac{U}{\epsilon}\log\frac{U}{\epsilon}\} \tag{8.39}$$

A suitable definition of the temperature, T, is given from,

$$\frac{1}{T} = \frac{dS}{dU} \tag{8.40}$$

Differentiating S we arrive at,

$$\frac{1}{T} = k\{\frac{1}{\epsilon}\log(1+\frac{U}{\epsilon}) - \frac{1}{\epsilon}\log(\frac{U}{\epsilon})\} \tag{8.41}$$

If we define $\beta = \frac{1}{kT}$ then we write,

$$\begin{aligned}
\beta\epsilon &= \log(1+\frac{U}{\epsilon}) - \log(\frac{U}{\epsilon}) && (8.42)\\
&= \log(1+\frac{\epsilon}{U}) && (8.43)\\
e^{\beta\epsilon} &= 1+\frac{\epsilon}{U} && (8.44)\\
U &= \frac{\epsilon}{e^{\beta\epsilon}-1} && (8.45)
\end{aligned}$$

If the energy ϵ is a function of the frequency, ν,

$$\epsilon = h\nu \tag{8.46}$$

Then average energy, U, is given by,

$$U = \frac{h\nu}{e^{\beta h\nu}-1}. \tag{8.47}$$

and the energy density u_ν can be written from the density of states,

$$u = \frac{8\pi\nu^2}{c^3}U = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{\beta h\nu}-1}. \tag{8.48}$$

gives Plank's Radiation Law.

8.1.3 Solution by canonical ensemble

The probability that a system is in energy state E_S is proportional to,

$$P_S = e^{-\beta E_S} \tag{8.49}$$

where $\beta = \frac{1}{k_B T}$ This means that after normalization we can write the probability as,

$$\begin{aligned}
P_S &= \frac{e^{-\beta E_S}}{\sum_S e^{\beta E_S}} && (8.50)\\
&= \frac{e^{-\beta E_S}}{Z} && (8.51)\\
Z &= \sum_S e^{-\beta E_S} && (8.52)
\end{aligned}$$

Where is a Z quantity known as the canonical ensemble. The average energy is then,

$$\bar{U} = \langle \epsilon \rangle = \frac{\sum_S E_S e^{-\beta E_S}}{Z}, \tag{8.53}$$

and the entropy as,

$$S = -\frac{\partial}{\partial T}(-k_B T \ln Z) \tag{8.54}$$

For continuous energies the sums become integrals,

$$P_S = \frac{\exp[-\beta E_S]}{\int_0^\infty \exp[-\beta E] dE} \tag{8.55}$$

$$\bar{U} = \frac{\int_0^\infty E \exp[-\beta E]}{\int_0^\infty \exp[-\beta E] dE} \tag{8.56}$$

If the energy is taken as the classical energy kT then we see that U quickly goes to infinity.

However if we take then energies as discrete, having values of,

$$E_s = s\epsilon, \text{ where } s = 0, 1, 2, 3, \cdots \tag{8.57}$$

$$\epsilon = h\nu \tag{8.58}$$

s is a number describing the number of photons in a cavity. The mode may be excited only in units of energy proportional to $h\nu$. (omitting the zero point energy $1/2h\nu$.)

Similar to the quantum mechanical oscillator with frequency ν, the energy eigenvalues are integral multiples of $h\nu$. Where s is understood as the number of photons contained in a mode. (in the QM oscillator s would be the quantum number.)

The quantity Z may be written more simply if the sum is evaluated,

$$Z = \sum_n e^{-\beta n h\nu} = 1 + e^{-\beta h\nu} + e^{-2\beta h\nu} + \ldots \tag{8.59}$$

$$= \frac{1}{1 - e^{-\beta h\nu}} \tag{8.60}$$

similarly we would like to evalute the sum in the average energy,

$$\langle \epsilon \rangle = \frac{\sum_n n h\nu e^{-\beta n h\nu}}{Z} \tag{8.61}$$

$$= h\nu \frac{\sum_n n e^{-\beta n h\nu}}{Z} \tag{8.62}$$

now we evaluate the sum as a derivative of the partition function Z.

$$\frac{d}{dq}\sum e^{-nq} = \sum n e^{-nq} = \frac{d}{dq}Z \tag{8.63}$$

$$\frac{d}{dq}\frac{1}{1 - e^{-q}} = \frac{-e^{-q}}{(1 - e^{-q})^2} \tag{8.64}$$

Combining eqns (8.26,8.62) we arrive at,

$$\langle\epsilon\rangle = (h\nu)\Big(\frac{-e^{-\beta h\nu}}{(1-e^{-\beta h\nu})^2}\Big)\Big(\frac{1}{1-e^{-\beta h\nu}}\Big)^{-1} \tag{8.65}$$

$$= (h\nu)\frac{-e^{-\beta h\nu}}{1-e^{-\beta h\nu}} \tag{8.66}$$

$$\bar{U} = \langle\epsilon\rangle = \frac{h\nu}{e^{\beta h\nu}-1} \tag{8.67}$$

Combine equations (8.5,8.26,8.67) to write the energy density as,

$$u(\nu)\mathbf{d}_\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{\beta h\nu}-1}d\nu, \tag{8.68}$$

$$u(\lambda)\mathbf{d}_\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\beta hc/\lambda}-1}d\lambda. \tag{8.69}$$

Plank calculated values of, $h = 6.55\times10^{-27}erg/sec$ and $k_B = 1.346\times10^{-16}erg/K$, while the standard values today are $h = 6.626\times10^{-34}J/s$ and $k_B = 1.38\times10^{-23}J/K$

The results of Plank's theory matches the experimental values, and solves the issue of the uv-catastrophe. Plank was not satisfied though, believing that his mathematical trick, did not have physical significance. History would prove Plank wrong on the physical significance and Plank won the Nobel Prize in 1918.

8.2 Einstein and the Photoelectric Effect

Einstein used Plank's concept of energy quantization to explain the photoelectric effect. He claimed that energy quantization is a fundamental property of E-M radiation.

If an Atom moves from $E_1 = n_1hf$ to energy $E_2 = n_2hf$, then the energy emitted by the atom will have an energy of,

$$E_2 - E_1 \tag{8.70}$$

Einstein stated that this energy has one state,

$$E_2 - E_1 = h\nu \tag{8.71}$$

where ν is the frequency of the Electromagnetic radiation emitted. This is a quantized packet of energy, he named a photon. Each photon has an energy defined by,

$$E = h\nu. \tag{8.72}$$

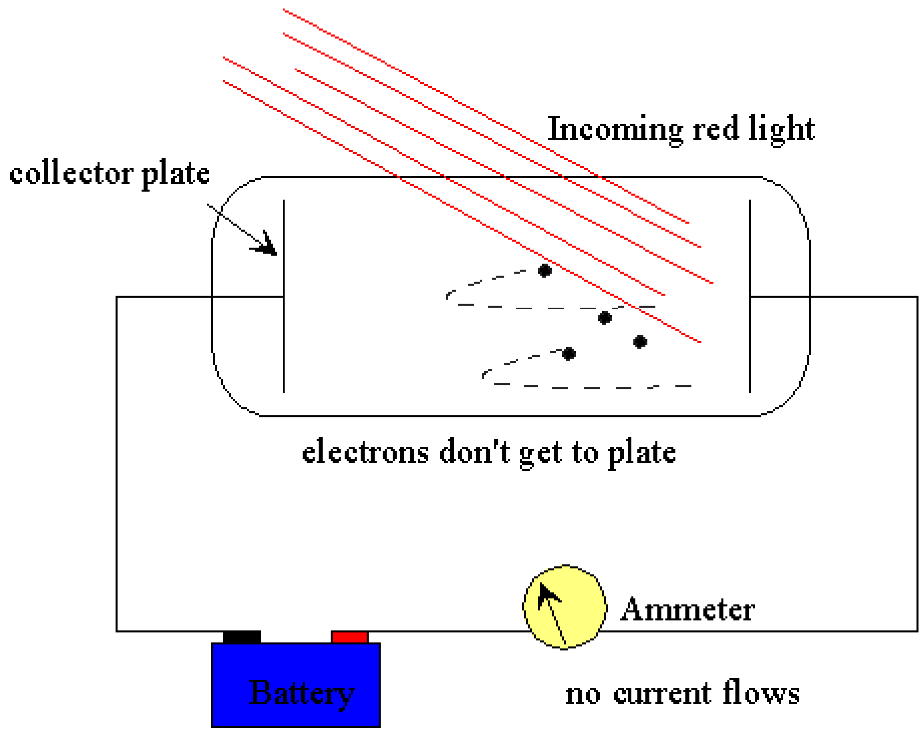
We notice 2 things;

1. Light being in discrete quanta implies that it acts like a particle. The intensity $\propto$ Number of photons.

2. Light has a frequency implying wave behavior, interference and diffraction.

8.2.1 The Photoelectric Effect

Certain Metallic materials are photosensitive, when light strikes the surface electrons are emitted.



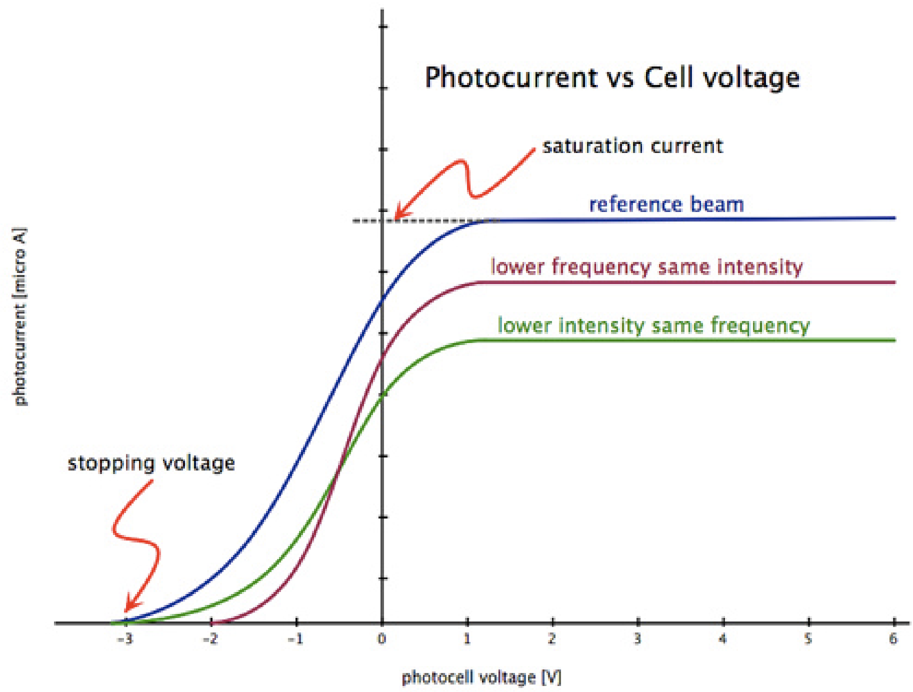
If these electrons are placed in an electric potential then the electrons will move creating a current.

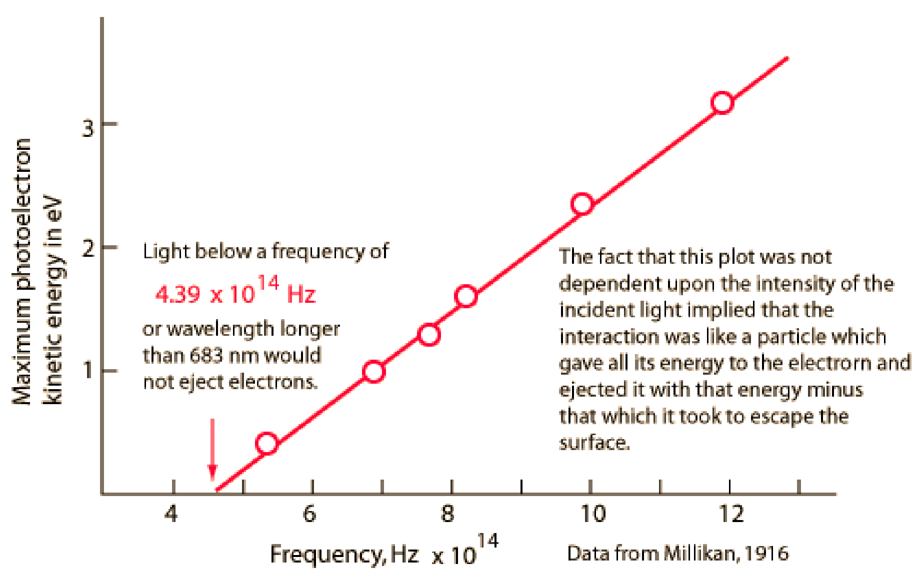
When the photocell is illuminated;

- Postive voltages the current does not vary w/ voltage only w/ intensity. $I_P \propto$ Intensity.
- $V < 0$ retarding voltage, electrons released from "cathode" must have enough K to over come the retarding voltage.
- Only electrons with $K > eV$ can produce a photo-current.
- At V_o (stopping potential) $I_P \rightarrow 0$

$$K_{max} = eV_o \tag{8.73}$$

- V_o same for all intensities
- No emissions for light $f < f_t$, f_t is the cutoff frequency.


Photocurrent vs Cell voltage
saturation current
reference beam
lower frequency same intensity
lower intensity same frequency
stopping voltage
photocurrent [micro A]
photocell voltage [V]
-3
-2
-1
0
1
2
3
4
5
6


Maximum photoelectron kinetic energy in eV
3
2
1
Light below a frequency of
4.39 x 10^14 Hz
or wavelength longer than 683 nm would not eject electrons.
The fact that this plot was not dependent upon the intensity of the incident light implied that the interaction was like a particle which gave all its energy to the electron and ejected it with that energy minus that which it took to escape the surface.
4
6
8
10
12
Frequency, Hz x 10^14
Data from Millikan, 1916

If light is in photons then the electron receives its energy from a photon. Intensity is the number of photons, but the energy absorbed by each electron is unchanged. One photon only has the energy to dislodge one electron, at a certain energy. We call ϕ the work function which is defined as the energy necessary to remove an electron from the cathode. The energy of the electrons emitted are given by Einstein's photo-electric equation,

$$(\frac{1}{2}mv^2)_{max} = hf - \phi = eV_0 \tag{8.74}$$

Where V_0 is the stopping potential.

We can see that if the stopping potential goes to 0. there is a minimum frequency that will eject an electron. This is called the threshold frequency, f_t.

$$\phi = hf_t = \frac{hc}{\lambda_t} \tag{8.75}$$

similarly we can define a threshold wavelength λ_t.

Homework 17
Photoelectric effect experiment

1. the apparatus for the h/e measurement is already set up by your instructor. Do not adjust.
2. Turn on the Hg light Source.
3. Turn on the h/e apparatus (photoelectric effect)
4. connect the voltmeter to the h/e appartatus.
5. Turn on the voltmeter.
6. Slowly rotate the h/e apparatus until the yellow light shines on the slot of the apparatus.
7. Only one color should fall on the photodiode window. Do not use the "RELATIVE TRANSMISSION" slide.
8. Place a yellow filter on the surface of the white reflective mask on the h/e apparatus.
9. Zero the h/e apparatus.
10. Read and record the output voltage on the voltmeter. This is the stopping voltage.
11. Repeat for each color, be sure to use the appropriate filter.

12. Plot V vs f, and find a slope, record values of the fit.
13. the vertical intercept is proportional to the work function $\phi = eV_o$, where V_o is the vertical intercept and $e = 1.6 \times 10^{-19}C$.
14. Compare ϕ to the correct value of $2.32 \times 10^{-19}J$.
15. The horizontal intercept gives the cutoff frequency, find the cutoff frequency f_o and compare to the correct value of $3.5 \times 10^{14}Hz$.
16. The slope gives the ratio h/e compare to the correct value of $0.414 \times 10^{-14}Js/C$

Example: Photon Energy

Calculate the photon energy for light of $\lambda = 400nm$ and $\lambda = 700nm$. $hc = 1240eVnm$

$$E = hf = \frac{hc}{\lambda} \tag{8.76}$$

$$E(\lambda = 400nm) = 3.1eV \tag{8.77}$$

$$E(\lambda = 700nm) = 1.77eV \tag{8.78}$$

Example

The threshold wavelength for potassium is 546nm (a) what is the work function of potassium? (b) What is the stopping potential when light of $\lambda = 400nm$ is incident on potassium?
SOLUTION:
(a)

$$\phi = hf_t = \frac{1240eVnm}{564nm} = 2.20eV \tag{8.79}$$

(b)

$$(\frac{1}{2}mv^2)_{max} = hf - \phi = eV_0 \tag{8.80}$$

$$eV_0 = hf - \phi = 3.10eV - 2.20eV = 0.90eV \tag{8.81}$$

Because the term 'e' or one electron appears on both sides of the equation, it may be canceled and the result is

$$V_0 = 0.90V \tag{8.82}$$

Einstein's model of light is consistent with all of the experimental results of the photoelectric effect, a phenomena not previously explained.

Homework 18
(2.1 in Tipler) Find the photon energy in joules and in electron volts for an electromagnetic wave inf the FM radio band of frequency 100MHz.

Homework 19
(2.7 in Tipler) The work function for tungsten is 4.58eV. (a) Find the threshold frequency and wavelength for the photoelectric effect. Find the stopping potential if the wavelength of incident light is (b) 200 nm and (c) 250 nm.

Homework 20
(2.9 in Tipler) The threshold wavelength for the photoelectric effect for silver is 262 nm. (a) Find the work function for silver. (b) Find the stopping potential if the incident radiation has a wavelength of 175 nm.

Chapter 9

Quantum mechanics II: The Atom

9.1 The Electron

- 1858 J. Plücker: for pressures less than $P < 10^{-6} atm$, emits invisible rays that carry charge (only visible when they strike the walls) These are termed cathode rays.
- 1879 William Crookes: showed
  - Cathode rays travel in straight lines and carry momentum
  - Bent my magnetic fields
  - negatively charged

J.J. Thomson's paper on cathode ray addresses the controversy of charge carriers through experimentation. There were two possibilities for the make up of cathode rays.

1. Cathode rays: are "due to some process in the aether" ie a wave.
2. Cathode rays: are electrified particles: rays are material, and the ray is the path of some charged particle

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Homework 21
What experiment(s) could be performed that would provoe or disprove these two conjectures?

9.1.1 J.J. Thomson discovers the electron

Thomson discusses 5 experiments: 4 performed by him and 1 that was previous to the paper.

Perrin's Experiment

Two cocentric cylinders separated by an insulator are placed in-front of a cathode ray. There is a hole in each cylinder so that the cathode rays can reach the inner cylinder.

- when the ray is not deflected by a magnetic field, the inner cylinder picks up a negative charge.
- when he ray is deflected there is no charging on the inner cylinder.

"This experiments proves that something charged with negative electricity is shot off of the cathode, ...and that this something is deflected by a magnet; it is open, however, to the objection that it does not prove that the cause of the electrification in the electroscope has anything to do with the cathode rays."

Thomson 1: Perrin Thomson's way

Now the apparatus has the same concentric cylinders but the are arranged such that the inner cylinder is only charged if the cathode ray is deflected such that it strikes the holes. Without deflection the ray does not hit the cylinders.

- When the cathode rays do not fall on the slit, there is no charging.
- When the rays hit the hole the charge changes

The conclusion is that the negative electrification follows the path of the cathode rays.

Thomson 2: Deflection by electric field

There is (was) an objection to the idea that cathode rays are a charged particle, in that under small electric fields there was no deflection of the cathode ray.

Thomson repeated the experiment showing the same result, but he doesn't stop there. He continues to examine this in more detail, proving that the absence of deflection is due to the conductivity passed to the gas by the cathode rays.

Observations

1. When the experiment is performed under a vacuum there is a deflection (after a time the deflection goes away due to the small amount of gas still present being turned into a conductor)
2. under low pressure and high potential, the cathode ray is deflected. when the medium breaks down(electric discharge), the ray jumps back to its undeflected position.
3. When the rays are deflected by an electrostatic field the phosphorescent band breaks into several bright bands separated by dark spaces. This is analogous to Birkland's magnetic spectra.

Thomson 3

This experiment uses the same apparatus as the previous, but it is used to examine the conductivity of the gas enclosed in the cathode tube. Then charge flow and deflection can be measured to determine some properties of the gas enclosed in the tube.

Thomson 4: magnetic deflection of the cathode rays in different gases

Thomson places a cathode tube between two Helmholtz coils this allows him to measure the radius ρ, of curvature on the cathode ray as related to the magnetic field H.

Radius of curvature and m/e

we can related the momentum of the charged particle to the field and the radius of curvature as,

$$\frac{mv}{e} = H\rho = I \tag{9.1}$$

The total charge that is transferred is,

$$Ne = Q \tag{9.2}$$

While the work done on the cathode is

$$W = \frac{1}{2}Nmv^2 \tag{9.3}$$

now we can write,

$$\frac{1}{2}\frac{m}{e}v^2 = \frac{W}{Q} \tag{9.4}$$

and,

$$\frac{m}{e} = \frac{I^2 Q}{2W} \tag{9.5}$$

W can be measured from the deflection of a galvanometer if the ray strikes a metal of known specific heat. This allowed Thomson measured $m/e \approx 0.5 \times 10^{-7}$.

Actual $m/e = 0.569 \times 10^{-7} emu/gram = 5.68 \times 10^{-12} kg/C$

Homework 22
PART I: m/e

1. Connect the poles of the $0 - 24$ Volts DC output to the Helmholtz coils, select current on the meter select (so we can read the current off of the upper scale of the power supply). Position the DC Voltage and the DC current knobs toward the middle. Keep the current near 1A, do not exceed 2A.

2. Connect the high voltage power supply to the e/m apparatus by, setting the knob for the 3A max on the "6" and connecting its two terminals to the heater on the e/m apparatus. Set voltage to 500V, connect the 0V and 500V (red high voltage pole) to the terminals on the electrods on the e/m apparatus. Adjust the 500V knob until the voltmeter reads a voltage between 100-120V. Do not excede 300.

3. Make sure the current adj knob on the e/m apparatus is not off.

4. Set the ammeter on 4A DC and place in the circuit with the low voltage supply.

5. Connect the voltmeter to the e/m apparatus, set the voltmeter to 400 V DC.

6. Have this all checked.

7. After the equipment is turned on, record, I, V, and measure and record the diameter of the circular path.

8. Calculate the magnetic field by, $B = 7.80 \times 10^{-4} I$

9. Calculate e/m from equation (**??**)

10. Calculate the accepted value for e/m from $e = 1.6 \times 10^{-19} C$ and $m = 9.11 \times 10^{-31} kg$, compare this to the measured value.

Thomson 5: m/e with E and B fields

For a electron passing through a constant magnetic field, we can calculate the velocity from the deflection,

$$F = qE \tag{9.6}$$

$$v_z = a_z t = \frac{-eE}{m_e} t \tag{9.7}$$

$$t = \frac{l}{v_x} \tag{9.8}$$

$$v_z = a_z t = \frac{-eE}{m_e} \frac{l}{v_x} \tag{9.9}$$

$$\tan\theta = \frac{v_z}{v_x} = -\frac{eEl}{m_e v_x^2} \tag{9.10}$$

Note: the $v \approx 0.1c$

Now if we add a transverse magnetic field we can select a velocity that passes through the combined E, and B fields without deflection,

$$F_z = -eE + ev_x B_y = 0 \tag{9.11}$$

$$v_x = \frac{E}{B} \tag{9.12}$$

$$\tag{9.13}$$

Now from the result previously,

$$\tan\theta = \frac{v_z}{v_x} = -\frac{eEl}{m_e v_x^2} \tag{9.14}$$

$$\frac{e}{m} = -\frac{E}{B^2 l}\tan\theta \tag{9.15}$$

Thomson's Conclusions

1. The charge carriers are the same for all gases
2. The mean free path of the charge carriers depend on the density of the gas.
3. Atoms are different aggregates of the same particles of which the electron is one
4. An Electric field is sufficient to remove the negatively charged particles from an atom.
5. Atoms are made of smaller particles
6. Calculating the stability of atoms is difficult because of the large number of particles involved.
7. velocity of cathode rays is proportional to the potential difference between the cathode and anode.
8. The material of which the cathode is manufactured is unimportant to the process.
9. "I can see no escape from the conclusion that they are charges of negative electricity carried by particles of matter."

9.1.2 Fundamental Charge: Robert Millikin

(notes from Modern Physics: for scientists and engineers by Taylor, Zafiratos and Dubson)

Thomson could measure the ratio of m/e but not m or e, he measured the ratio from the charges iteration with magnetic and electric fields

$$m\vec{a} = -e(\vec{E} + \vec{v} \times \vec{B}) \tag{9.16}$$

Robert Millikin was able measure the mass of a oil drop for which the mass was known and measure the charge on that oil drop.

Experimental method

1. Spray a fine mist into the region between 2 charged plates. The drops fall reaching terminal speed with the weight of the drop balanced by the viscous drag of air.

2. Switch on an electric field, some drops move down more rapidly, some drops move upward.

The conclusion is some oil drops have acquired positive or negative charges.

9.1.3 Measuring M by terminal speed

The terminal velocity may be calculated from Stoke's Law,

$$F = 6\pi\eta r v \tag{9.17}$$

$$v = \frac{F}{6\pi r\eta} \tag{9.18}$$

where, r =radius of sphere, η= viscosity of gas, $F = Mg = 4/3\pi r^3\rho g$= weight of oil drop. Now we write,

$$r^2\frac{18}{4}\frac{\eta v_1}{\rho g} \tag{9.19}$$

Since ρ, g, η are known, a measurement of the velocity gives the radius and the radius gives us the mass M.

After the inclusion of an electric field,

$$qE - mg = 6\pi\eta r v_2 \tag{9.20}$$

$$v_2 = \frac{qE - mg}{6\pi\eta r} \tag{9.21}$$

$$\frac{v_2}{v_2} = \frac{qE - mg}{mg} = \frac{qE}{mg} - 1 \tag{9.22}$$

Millikin also noticed that occasionally a drop would suddenly move up or down indication a change in charge. Millikin theorized that this was due to gaining electrons from the ionized air in the chamber. In order to test this he ionized the air by exposure to x-rays. In this way he studied the change in charge.

Subsequently, he measured that all changes in charge were integer multiples of $e = 1.6 \times 10^{-19} C$. This is the basic unit of charge. The charge of an electron.

Millikin also noticed when there was a near vacuum in the container x-rays changed that charge but only to the positive, Leading MIllikin to correctly conclude that the x-rays are knocking electrons out of the oil drops.

2 important results,

1. The charge of one electron is $q = -e$, this leads to the mass of an electron being 1/2000 Hydrogen

2. All charge come in multitudes of e.

9.2 The Nucleus: Rutherford

(notes from Modern Physics: for scientists and engineers by Taylor, Zafiratos and Dubson)

The great question after Thomson discovered the electron was, **what is the structure of the atom?** Thomson gave an opinion; **Thomson Model:** Electrons embedded in a sphere of uniform positive charge (blueberries in a muffin).

To test this charged particles were used to bombard a target of a metallic foil. Helium nuclei known as α-particles were used. (1909 Rutherford).

If α particles strike a thin layer of matter most pass though suffering only small deflections. This implies that atoms are mostly empty space. Deflections are caused by the electric fields of atoms. Most deflections are small, however a few particles were deflected through large angles $\sim 90°$. Thompson believed that large angles were the result of many deflections. Rutherford, however showed that the probability of striking one atom was small, and many atoms exceedingly small. This implies that the electric field of an atom must be greater than previously believed.

The electric field may be calculated from,

$$E = \frac{kq}{r^2} \tag{9.23}$$

and if k is a constant and q cannot change the only way to increase the electric field is to reduce the radius r.

In light of the need for a decreased radius, Rutherford proposed an atom with all of the positive or negative charge concentrated into a tiny massive nucleus, with a cloud of opposite charge outside of the nucleus. most *alpha* particles pass far from the nucleus those that get close are scattered through large angles. These deflections can be explained by coulomb forces.

Rutherford calculated;

- The trajectory of an *alpha*-particle through different angles

- Predicted number of deflections through different angles
- Predicted how this number would vary with foil thickness

These were verified by Gieger and Marsden 1913 two scientists working with Rutherford.

9.2.1 The Rutherford Formula

[INSERT PICTURE]

From coulomb's law we find the force on an α-particle of mass m and charge $q = 2e$, from a nucleus of charge $Q = Ze$ can be calculated as,

$$\begin{aligned} F &= \frac{kqQ}{r^2} && (9.24) \\ &= \frac{2Zke^2}{r^2} && (9.25) \end{aligned}$$

where $k = 8.99 \times 10^9 Nm/C^2$.

We assume that the collision is perfectly elastic. Similarly the mass of the nucleas is so much greater than that of the α-particle, thus we can assume that the magnitude of the momentem does not change, allowing us to write the change in the momentum of the α-particle as,

$$\Delta p = 2p_i \sin\frac{\theta}{2} \tag{9.26}$$

From Newton's second law we write,

$$\Delta p = \int_{-\infty}^{\infty} F dt \tag{9.27}$$

Now we choose x in the direction of the change in momentum,

$$\begin{aligned} \Delta p &= \int_{-\infty}^{\infty} F_x dt && (9.28) \\ &= \int_{-\infty}^{\infty} \frac{2Ze^2k}{r^2}\cos\phi dt && (9.29) \\ & && (9.30) \end{aligned}$$

We can write the initial angular momentum as $L = p_i b$, and $L = mr^2\omega = mr^2\frac{d\phi}{dt}$ this allows us to calculate $\omega = \frac{d\phi}{dt}$ as,

$$\frac{d\phi}{dt} = \frac{p_i b}{mr^2} \tag{9.31}$$

Now we calculate the change in momentum as,

$$\begin{aligned}\Delta p &= \int_{-\infty}^{\infty} \frac{2Ze^2k}{r^2}\cos\phi dt, && (9.32)\\ &= 2kZe^2 \int \frac{\cos\phi}{r^2}\frac{dt}{d\phi}d\phi, && (9.33)\\ &= \frac{2kZe^2}{bp_i}\int_{\phi_i}^{\phi_f}\cos\phi d\phi, && (9.34)\\ &= \frac{2kZe^2}{bp_i}\big(\sin\phi_f - \sin\phi_i\big) && (9.35)\end{aligned}$$

Since $\phi_f = -\phi_i$, and $\theta + 2\phi_f = 180°$, $\phi = 90° - \theta/2$ and $\sin\phi_f = \cos(\theta/2)$.

Now we write Δp as,

$$\begin{aligned}\Delta p &= \frac{4kZe^2m}{bp_i}\cos(\frac{\theta}{2}) && (9.36)\\ &= 2p_i\sin(\frac{\theta}{2}) && (9.37)\\ & && (9.38)\end{aligned}$$

This leads to the relationship between the impact parameter b and the scattering angle θ.

$$b = \frac{Zke^2}{E\tan(\frac{\theta}{2})}. \tag{9.39}$$

where $E = p_i^2/2m$. This equation can be understood as particles with an impact parameter $b' < b$ will scatter at an angle $\theta' > \theta$. [INCLUDE PICTURE]

We can now calculate the number of particles scattered between an angle θ and $\theta + d\theta$.

If the beam of α-particles has a crossectional area of A. The proportion of atoms scattered by θ or more is $\pi b^2/A$, so that the number of particles scattered is,

$$N_s = N\frac{\pi b^2}{A} \tag{9.40}$$

Multiply this by the number of target atoms illuminated by the beam, where the foil has a thickness of t and contains n nuclei per unit volume.

$$n_t = nAt \tag{9.41}$$

$$N_{sc}(\theta' > \theta) = (\frac{N\pi b^2}{A})(nAt) = \pi Nntb^2 \tag{9.42}$$

Now we substitute in b in terms of θ,

$$N_{sc} = \pi Nnt\Big(\frac{Zke^2}{E\tan(\frac{\theta}{2})}\Big)^2 \tag{9.43}$$

Now we differentiate with respect to θ,

$$dN_{sc}(\theta \to \theta + d\theta) = Nnt\Big(\frac{Zke^2}{E}\Big)^2 \frac{\cos(\frac{\theta}{2})}{\sin^3(\frac{\theta}{2})} d\theta \tag{9.44}$$

These particles are scattered across a ring of area,

$$dA = (2\pi s \sin\theta)(sd\theta) \tag{9.45}$$

then the number of particles scattered between $\theta \to \theta + d\theta$ over unit area is,

$$n_{sc}(\theta) = \frac{dN_{sc}}{dA} = \frac{Nnt\Big(\frac{Zke^2}{E}\Big)^2 \frac{\cos(\frac{\theta}{2})}{\sin^3(\frac{\theta}{2})} d\theta}{2\pi s^2 \sin\theta d\theta}, \tag{9.46}$$

$$= \frac{Nnt}{4s^2}\Big(\frac{Zke^2}{E}\Big)^2 \frac{1}{\sin^4(\theta/2)} \tag{9.47}$$

where, $\sin\theta = 2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})$. Eq (9.47) is known as the Rutherford formula. N original number of α-particles, and s^2 is the square of the distance to detector.

1. $n_{sc}(\theta) \propto t$ thickness
2. $n_{sc}(\theta) \propto (Ze)^2$ nuclear charge squared
3. $n_{sc}(\theta) \propto 1/E^2$ the incident energy squared
4. $n_{sc}(\theta) \propto 1/\sin^4(\theta/2)$

The fact that the coulomb force always holds gives an upper bound on the size of a nucleus.

Example: upper bound on nucleus size

assume an α-particle with energy 7.7Mev is fired at gold foil (Z=79), find the upper limit on nuclear size.

The distance of the particle to the nuclear center is always greater than the radius of the nucleus, $r > R$. The minimum value of r will occur for the case of a head on collision, at the point of minimum r, the particle has an instanaeous minimum (as the particle turns around) in the kinetic energy (K=0). Thus its potential energy here is equal to its total energy,

$$U = \frac{2Zke^2}{r_{min}} = E = 7.7Mev \tag{9.48}$$

now the r_{min} is now,

$$r_{min} = \frac{2kZe^2}{E} \tag{9.49}$$

since $R < r$ for all orbits,

$$R \lesssim \frac{2kZe^2}{E} \tag{9.50}$$

we can write $ke^2 = 1.44MeVfm$, this allows us to calculate, $R \lesssim 30fm$

We can see that as the energy increases, the rutherford formula breaks down as the minimum value of r approaches the actual value of R. The energy at which the rutherford fromula first breaks down is

$$E \approx \frac{2kZe^2}{R} \tag{9.51}$$

Homework 23
(3.49 in Taylor) (a) If the Rutherford formula is found to be correct at all angles when 15-MeV alpha particles are fired at silver foil (Z=47), what can you say about the radius of the silver nucleus? (b) Aluminum has atomic number Z=13 and a nuclear radius $R_{Al} \approx 4fm$. If one were to bombard aluminum foil with alpha particles and slowly increase their energy, at about what energy would you expect the Rutherford formula to break down? [You can make the estimate a bit more realistic by taking R to be $R_{Al} + R_{He}$ where R_{He} is the alpha particles radius (about 2 fm).

9.3 The Bohr Model

By 1870 the atomic spectra from emission and absorption was comonly used to identify elements. There was however no satisfactory explaination for the spectra from classical physics.

By 1885 the four visible lines of the hydrogen molecule had been accurately measured, and swiss schoolteacher Johann Balmer had shown that these wavelengths fit the heuristic formula,

$$\frac{1}{\lambda} = R\Big(\frac{1}{4} - \frac{1}{n^2}\Big) \tag{9.52}$$

where $n = 3, 4, 5, 6$ for the visible lines, and R is a constant of value $R = 0.0110m^{-1}$. This was generalized to,

$$\frac{1}{\lambda} = R\Big(\frac{1}{n'^2} - \frac{1}{n^2}\Big) \tag{9.53}$$

This equation is known as the Rydberg formula and describes all wavelengths in the hydrogen spectra.

A satisfacory model would have to wait until 1913 when Niels Bohr published "On the Constitution of Atoms and Molecules". Bohr postulated that electrons are bound in stable energy levels, and energy is only emitted when an electron passes between levels. From this Bohr was able to predict the Rydberg formula.

Bohr's model of the atom was a refinement of Rutherford's model. This model is planetary, with the nucleus functioning as the sun and electrons as the planets. Under classical electro-magenetic theory if a particle is accelerating, then it is emitting electromagnetic waves. When a particle is radiating it is losing energy and the orbit will decay at a rate of $10^{-11}s$. Thus under classical electro-magenetic theory Rutherford's model is unstable.

Bohr proposed as a refinement, only a certain discrete set of orbits are possible. This leads to two implications,

- the energies of such orbits are discrete,
- the electron only radiates when passing between discrete orbits.

9.3.1 The Bohr radius

Bohr assumed the orbits where close to classical orbits, then from the coulomb force we find,

$$F = \frac{ke^2}{r^2} \tag{9.54}$$

$$m\frac{v^2}{r} = \frac{ke^2}{r^2} \tag{9.55}$$

classically there is no restriction on v, or r which may range between 0 and ∞. we can write the potential and kinetic energy as,

$$K = \frac{1}{2}mv^2 \tag{9.56}$$

$$U = -\frac{ke^2}{r} \tag{9.57}$$

At infinity $U = 0$ this leads to the kinetic energy and total energy being,

$$K = -\frac{1}{2}U \tag{9.58}$$

$$E = K + U = \frac{1}{2}U = -\frac{1}{2}\frac{ke^2}{r} \tag{9.59}$$

This result is commonly known as the viral theorem.

Bohr theorized that the orbits are discrete, meaning that like energies the angular momentum is quantized.

Bohr then postulated that the angular momentum, L is quantized and proportional to plank's constant.

$$L = \frac{h}{2\pi}, 2\frac{h}{2\pi}, 3\frac{h}{2\pi}, 4\frac{h}{2\pi}, \ldots \tag{9.60}$$

$$L = n\hbar \tag{9.61}$$

where $\hbar = h/2\pi = 1.054 \times 10^{-34} Js$. Now can write the angular momentum as,

$$mvr = n\hbar \tag{9.62}$$
$$v = \frac{n\hbar}{mr}. \tag{9.63}$$

Calculate the radius of the orbit, from the conservation of energy,

$$K = -\frac{1}{2}U, \tag{9.64}$$
$$\frac{1}{2}mv^2 = \frac{1}{2}\frac{ke^2}{r}, \tag{9.65}$$
$$m(\frac{n\hbar}{mr})^2 = \frac{ke^2}{r}, \tag{9.66}$$
$$r = n^2\frac{\hbar^2}{ke^2m} = n^2a_0. \tag{9.67}$$

where $a_0 = \frac{\hbar^2}{mke^2} = 0.0529nm$ is the Bohr radius or the radius of the hydrogen atom in its ground state. This is not exact because electron positions are merely probabilty densities, but it is very close to the average position of the electron.

9.3.2 The energy of the Bohr atom

Knowing the possible radii we can calculate the possible energies,

$$E = -\frac{ke^2}{2r} \tag{9.68}$$
$$= -\frac{ke^2}{2a_0}\frac{1}{n^2}, \text{ where } n = 1, 2, 3\ldots \tag{9.69}$$

Thus the possible energies are quantized, and the energy emitted when dropping from one energy level to another is,

$$E_\gamma = E_n - E'_n \tag{9.70}$$
$$= \frac{ke^2}{2a_0}\Big(\frac{1}{n'^2} - \frac{1}{n^2}\Big) \tag{9.71}$$
$$= hcR\Big(\frac{1}{n'^2} - \frac{1}{n^2}\Big) \tag{9.72}$$

which is the result given by Rydberg.

Now we calculate R from the values here,

$$R = \frac{ke^2}{2a_o hc}, \tag{9.73}$$
$$= \frac{\alpha}{4\pi a_0}, \tag{9.74}$$
$$= \frac{1.44eV\dot{n}m}{(2)(0.0529nm)(1240eV\dot{n}m)} \tag{9.75}$$
$$= 0.0110m^{-1} \tag{9.76}$$

note $\alpha = 1/137$ is a constant known as the fine structure constant. One finds the value found from Bohr's theory for R is equivalent to the experimental value.

We define the Rydberg energy as,

$$E_R = hcR \tag{9.77}$$
$$= \frac{ke^2}{2a_0} \tag{9.78}$$
$$= \frac{m(ke^2)^2}{2\hbar^2} \tag{9.79}$$
$$= 13.6eV \tag{9.80}$$

when $n = 1$, now we can write the energies of each orbit as,

$$E_n = -\frac{E_R}{n^2} \tag{9.81}$$

The lowest possible energy corresponds to $n = 1$ this is known as the ground state, states with higher energies are known as excited states.

- The ground state energy corresponds to the energy required to remove an electron entirely from an atom (in this case from hydrogen)
- excellent agreement with experiment
- $r = a_0 = 0.0529nm$ average radius of electron orbit
- Generalized Bohr's theory gives a good order of magnitude estimate of other atomic radii.

[INSERT PICTURE]

9.3.3 Generalizing the Bohr Model

we now have nuclear charges of Ze thus we write the force as,

$$F = \frac{Zke^2}{r} \tag{9.82}$$

This allows us to calculate the radius as,

$$\begin{aligned} r &= n^2 \frac{\hbar^2}{Zke^2 m} && (9.83) \\ &= n^2 \frac{a_0}{Z} && (9.84) \end{aligned}$$

And we find that the energy of the nth orbit is given as,

$$E_n = -Z^2 \frac{ke^2}{2a_0} \frac{1}{n^2} = -Z^2 \frac{E_R}{n^2} \tag{9.85}$$

For a He^+ ion the photons emitted are given by,

$$E_\gamma = 4E_R \Big(\frac{1}{n'^2} - \frac{1}{n^2} \Big) \tag{9.86}$$

This has been observed in solar spectra.

We should note that because there real center of mass of the atom is not coincident with the center of the nucleus, we can correct the Rydberg energy by,

$$E_R = \frac{\mu(ke^2)^2}{2\hbar^2} \tag{9.87}$$

where μ is the reduced mass,

$$\mu = \frac{m}{1 + m/m_n} \tag{9.88}$$

m is the mass of an electron and m_n is the mass of a nucleus.

Homework 24
(5.11 in Taylor) Find the range of wavelengths in the Balmer series of hydrogen. Does the Balmer series lie completely in the visible region of the spectrum? If not, what other regions does it include?

Homework 25
(5.13 in Taylor) The negative muon is a subatomic particle with the same charge as the electron but a mass that is about 207 times greater: $m_\mu \approx 207 m_e$. A muon can be captured by a proton to form a "muonic hydrogen atom", with energy and radius given by the Bohr model, except that m_e must be replaced by m_μ. (a) What are the radius and energy of the first Bohr orbit in muonic hydrogen? (b) [not assigned] What is the wavelength of the Lyman α line in muon hydrogen? What sort of electromagnetic radiation is this? (Visible? IR? etc.) Treat the proton as fixed.

9.4 The Neutron

Chapter 10

Quantum mechanics III: Elements of Quantum Theory

10.1 Compton Scattering

When a beam of light is fired at a system of charges, some of the beam is scattered. The classical theory says the oscillation electric field of the incident light causes the charges to oscillate, and the oscillating charges then radiate secondary radiation in various directions. The frequency f of the scattered waves must be the same as that of the oscillating charges, which must be the same as the incident frequency f_0. Classically $f = f_0$.

Starting in 1912 reports of High frequency x-rays being scattered of electrons, where the scattered frequency was less than the incident frequency $f < f_0$.

In 1923 Arthur Compton argued that if light is quantized, one should expect $f < f_0$. Since photons carry energy they should also carry momentum. From the Pythagorean energy in special relativity,

$$E^2 = (pc)^2 + (mc^2)^2, \tag{10.1}$$

we calculated the energy of a massless object as,

$$E = pc. \tag{10.2}$$

From the photoelectric effect $E = h\nu$ allowing us the write the momentum as,

$$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} \tag{10.3}$$

Thus in a collision with a stationary electron there is a transfer of energy and momentum, causing a reduction in the energy of a photon.

Naturally, the electrons are not purely stationary, the energies of outer shell electrons are on the order of 1eV. While, the energies of an x-ray photon correspond to wavelengths between $0.001nm < \lambda < 1nm$, or energies between, $1.2MeV < E < 1.240Mev$. The energy of the electron is small enough to neglect in terms of calculation. [INSERT PICTURE] We will calculate the results using an elastic collision where, energy and momentum are conserved.

$$\begin{aligned} E_e + E_1 &= mc^2 + E_0 && (10.4) \\ \vec{p}_e + \vec{p}_1 &= \vec{p}_0 && (10.5) \\ E_e &= mc^2 + E_0 - E_1 && (10.6) \end{aligned}$$

Where E_e, p_e is the energy and momentum of the electron after the collision, E_0, p_0 is the energy and momentum of the photon before the collision and E_1, p_1 is the momentum and energy of the photon after the collision. We treat the electron relativistically by writing $E_0 = p_o c$ and $E_1 = p_1 c$. Now we write the conservation of energy as,

$$\begin{aligned} E_e &= mc^2 + E_0 - E_1 && (10.7) \\ \sqrt{p_e^2 c^2 + (mc^2)^2} &= mc^2 + p_0 c - pc && (10.8) \\ \sqrt{p_e^2 + (mc)^2} &= mc + p_0 - p && (10.9) \\ & && (10.10) \end{aligned}$$

Now we wish to replace p_e with a function of the photon momentum, by using the conservation of momentum equation.

$$\begin{aligned} \vec{p}_e &= \vec{p}_0 - \vec{p} && (10.11) \\ p_e^2 &= \vec{p}_e \cdot \vec{p}_e = (\vec{p}_0 - \vec{p}) \cdot (\vec{p}_0 - \vec{p}) && (10.12) \\ &= p_0^2 + p^2 - 2\vec{p}_0 \cdot \vec{p} && (10.13) \\ &= p_0^2 + p^2 - 2p_0 p \cos\theta && (10.14) \\ & && (10.15) \end{aligned}$$

Now,

$$\begin{aligned} \sqrt{p_e^2 + (mc)^2} &= mc + p_0 - p && (10.16) \\ p_e^2 + (mc)^2 &= (mc + p_0 - p)^2 && (10.17) \\ p_0^2 + p^2 - 2p_0 p \cos\theta + (mc)^2 &= (mc)^2 + 2p_0 mc - 2pmc - 2p_0 p + p_0^2 + p^2 && (10.18) \\ mc(p_0 - p) &= p_0 p(1 - \cos\theta) && (10.19) \\ \frac{1}{p_0} - \frac{1}{p} = \frac{1}{mc}(1 - \cos\theta) & && (10.20) \end{aligned}$$

since $p = h\nu/c$, we write,

$$\frac{1}{\nu_1} - \frac{1}{\nu_0} = \frac{h}{mc^2}(1 - \cos\theta) \qquad (10.21)$$

or in terms of the wavelength we write,

$$\Delta\lambda = \lambda_1 - \lambda_0 = \frac{h}{mc}(1 - \cos\theta) \tag{10.22}$$

We observe that the shift depends on the angle. We can replace the term h/mc by the Compton wavelength $\lambda_C = h/mc = 0.00243nm$.

$$\lambda_1 - \lambda_0 = \lambda_C(1 - \cos\theta) \tag{10.23}$$

- λ_{max} occurs at $\theta = 180°$

$$Delta\lambda_{max} = 2\lambda_c = 4.86 \times 10^{-3}nm \tag{10.24}$$

- Since this is the max charge, λ's that are several thousand times larger are hard to measure the Compton shift.
- Compton effect negligible for UV, visible, IR, only noticeable for x-rays, γ-rays
- Einstein and Compton
  - light is a particle
  - dual nature = wave-particle duality

10.1.1 Example: Compton Scattering

Calculate the percentage change in λ observed for Compton scattering of 20keV photons at $\theta = 60°$.

$$\begin{aligned} \lambda_1 - \lambda_0 &= \lambda_C(1 - \cos\theta) = 1.22pm & (10.25)\\ \lambda_0 &= \frac{hc}{E} = 0.062m = 62pm & (10.26)\\ \frac{\Delta\lambda}{\lambda_1} \times 100\% &= 1.97\% & (10.27) \end{aligned}$$

Now calculate the maximum precent change for the $\lambda = 0.0711nm$ x-rays used by Compton in his scattering experiments off of graphite electrons. The maximum corresponds to the maximum change in kinetic energy of the electron, or the greatest change in wavelength where $\theta = 180°$.

$$\begin{aligned} \Delta\lambda &= \lambda_C(1 - \cos\theta) = 0.00486nm & (10.28)\\ \frac{\Delta\lambda}{\lambda_1} \times 100\% &= \frac{0.00486nm}{0.0711nm} = 6.8\% \approx 7\% & (10.29) \end{aligned}$$

Calculate the maximum percent change for a visible light at $\lambda = 400nm$,

$$\begin{aligned} \Delta\lambda &= \lambda_C(1 - \cos\theta) = 0.00486nm & (10.30)\\ \frac{\Delta\lambda}{\lambda_1} \times 100\% &= \frac{0.00486nm}{400nm} \approx 1 \times 10^{-3}\% & (10.31) \end{aligned}$$

Homework 26
(Tipler) Compton used photons of $\lambda = 0.0711nm$ (a) What is the energy of these photons? (b) What the λ of the photons scattered at $\theta = 180°$? (c) What is the energy of the scattered photons?

Homework 27
(4.31 in Taylor) If the maximum kinetic energy given to the electronic a Compton scattering experiment is 10keV, what is the wavelength of the incident x-rays.

10.2 de Broglie

Compton showed that a photon displays both the properties of a wave $E = hf$ and the properties of a particle $p = h/\lambda$

de Broglie postulated that material particles have a similar particle-wave duality, and as such they should have a wavelength defined by

$$\lambda = \frac{h}{p}, \tag{10.32}$$

10.2.1 Example: de Broglie wavelength

Find the de Broglie wavelength of a particle of mass $10^{-6}g$ moving at a speed of $v = 10^{-6}m/s$.

$$\lambda = \frac{h}{p} = \frac{h}{mv} \tag{10.33}$$

$$= \frac{6.63 \times 10^{-34} Js}{(10^{-9}kg)(10^{-6}m/s)} = 6.63 \times 10^{-19}m \tag{10.34}$$

Since the wavelength is much smaller han any possible obstacles diffraction or interference of such waves cannot be observed.

Two important experiments in 1927 established the existence of the wave properties of electrons

1. Davisson, Germer Electron scattering from a nickel target. The intensity of scattered electrons varied with angle, showing the results of diffraction.

2. G.P. Thompson (Son of J.J.) Observed electron diffraction in the transmission through metal foils

The calculation of the wavelength from observed diffraction is the same as the de Broglie wavelength.

Diffraction of electrons leads to the electron microscopes.

Homework 28
(Tipler) A proton is moving at $v = 0.003c$, find its de Broglie wavelength.

10.2.2 Quantization of angular momentum

de Broglie argued that if electrons were a wave then the angular momentum would be quantized in multiples of $\hbar$. A wave can then fit into a circular path, if the circumference can accommodate and integral number of λs.

$$2\pi r = n\lambda \; n = 1, 2, 3, \ldots \tag{10.35}$$
$$\lambda = \frac{h}{p} \tag{10.36}$$
$$2\pi r = n\frac{h}{p} \tag{10.37}$$
$$rp = \frac{nh}{2\pi} \tag{10.38}$$
$$L = n\hbar \; n = 1, 2, 3, \ldots \tag{10.39}$$
$$\tag{10.40}$$

Which is the same result as Bohr assumptions in his model of the hydrogen atom.

10.3 Wave-particle duality

Light and matter both exhibit qualities of both waves and particles. Which is true?

de Broglie believed the relationship was such that waves can behave like particles and particles like waves.

10.3.1 Young's Double slit Experiment

Imagine two slits.

Particles striking 2 slits.

1. come in lumps
2. probability of hitting is the sum of the probability from each slit.

waves striking 2 slits

1. continuous
2. measure intensity
3. Intensity from 2 slits not the sum of the intensity from each slit

4. shows interference.

electrons striking 2 slits

1. come in lumps

2. measure probability of hitting

3. probability from 2 slits not the sum of the probability from each slit

4. shows interference.

We observe this interference pattern even for 1 electron, meaning it interferes with itself. The experiments show $p = h/\lambda$.

A quantum particle can be described as a 'wave' that interferes with itself. In fact, we cannot predict where the electron strikes.

This is a fundamental shift to a probabilistic dynamics rather than a deterministic one.

We can determine a distribution of probability, which tells us the chance that a quantum particle hits a particular spot. In the Schroedinger description of quantum mechanics this probability amplitude is called the wave function ψ.

10.4 Probability Waves

The wave function for a matter wave is designated by the greek letter psi ψ. The motion of a single electron is described by the wave function ψ, which is a solution to the Schrödinger equation and contains imaginary numbers. The wave functions are not necessarily real. The probability then of finding an electron in some region of space is $|\psi|^2$. We call this the probability density,

$$P(x) = |\psi|^2 \tag{10.41}$$

1926 Edwin Schroedinger (Austrian) Presented a general equation that describes the deBroglie matter waves,

$$(K + U)\psi = E\psi \tag{10.42}$$

K, U, and E are operators that relate to the kinetic energy potential and total energy, using calculus we can calculate ψ.

For a hydrogen atom in a ground state $n = 1$ the maximum of $|\psi|^2$ corresponds to the Bohr radius.

10.4.1 Example: Probability density

A classical particle moves back and forth with constant speed between 2 impeneratble walls at $x = 0$ and $x = 8cm$. (a) what the probability density $P(x)$? (b) What is $P(x = 2cm)$? (c) What is $P(x = 3 \to 3.4cm)$?
(a)

$$P(x) = P_0 \; 0 < x < 8cm \tag{10.43}$$

$$\int_0^\infty P_0 dx = 1 \tag{10.44}$$

The probability of finding the particle in the interval dx at point x_1 or at point x_2 is the sum of the separate probabilities $P(x_1)dx + P(x_2)dx$. Since the particle must certainly be somewhere, the sum of the probabilities over all possible values must equal 1.

$$\int_0^\infty P(x)dx = \int_0^{8cm} P_0 dx = 1 \tag{10.45}$$

$$p_0 = \frac{1}{8cm} \tag{10.46}$$

(b) Since $dx = 0$ The probability of finding a particle at the point $x = 2cm$ is 0.

(c) Since the probability density is constant the probability of a particle being in the range Δx is $P_0 \Delta x$. The probability of the particle being in the region $3.0cm < x < 3.4cm$ is thus,

$$P_0 \Delta x = (\frac{1}{8cm})0.4cm = 0.05 \tag{10.47}$$

10.5 Electron wave packets

A Harmonic wave on a spring is represented by the wave function

$$y(x,t) = A \sin(kx - \omega t) \tag{10.48}$$

Where $k = 2\pi/\lambda$ is known as the wave number, and the angular frequency is $\omega = 2\pi f$. The velocity of the wave then is given by,

$$v = f\lambda = (\frac{\omega}{2\pi})(\frac{2\pi}{k}) = \frac{\omega}{k} \tag{10.49}$$

ω, k have no limit in space-time.

One electron that is localized in space must be explained by a wave packet. In order to generate a wave packet that is localized in space we need a group of harmonic waves containing a continuous distribution of k, ω.

Consider a grouping of 2 waves,

$$\psi(x,t) = A_0 \sin(k_1 x - \omega_1 t) + A_0 \sin(k_2 x - \omega_2 t) \tag{10.50}$$

using,

$$\sin\theta_1 + \sin\theta_2 = 2\cos(\frac{\theta_1 - \theta_2}{2})\sin(\frac{\theta_1 + \theta_2}{2}) \tag{10.51}$$

We can write the wave as,

$$\psi(x,t) = 2A_0 \cos(\frac{1}{2}\Delta kx - \frac{1}{2}\Delta\omega xt)\sin(\frac{1}{2}k_{ave}x - \frac{1}{2}\omega_{ave}xt) \tag{10.52}$$

This wave describes the superposition of two waves with velocitie $v = \frac{\Delta\omega}{\Delta k}$ and $v = \frac{\omega_{ave}}{k_{ave}}$. If we place x_1 at the position where both waves are zero and x_2 at the next such point, then we know that,

$$\frac{1}{2}\Delta kx_1 - \frac{1}{2}\Delta kx_2 = \pi \tag{10.53}$$

which can be rewritten as,

$$\Delta k\Delta x = 2\pi \tag{10.54}$$

In General,

$$\psi(x,t) = \sum_i A_i \sin(k_i x - \omega_i t) \tag{10.55}$$

- To solve A_i's requires Fourier analysis
- Requires A continuous distribution of waves
- Replace A_i with $A(k)dk$

The group velocity of the wave packet becomes

$$v_g = \frac{d\omega}{dk} \tag{10.56}$$

We need to find $\omega(k)$, Starting with the Energy and momentum,

$$E = hf = h\frac{\omega}{2\pi} = \hbar\omega \tag{10.57}$$

$$p = \frac{h}{\lambda} = \frac{hk}{2\pi} = \hbar k \tag{10.58}$$

Now since we can write the kinetic energy as,

$$E = \frac{p^2}{2m} \tag{10.59}$$

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} \tag{10.60}$$

$$\omega() = \frac{\hbar k^2}{2m} \tag{10.61}$$

Now we find the group velocity as,

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}\left(\frac{\hbar k^2}{2m}\right) = \frac{\hbar k}{2m} = \frac{p}{m} = v \tag{10.62}$$

where v is the velocity of the electron.

The phase velocity of the individual waves inside the wave packet is equal to,

$$v_p = \frac{\omega}{k} = \frac{\hbar\omega}{\hbar k} = \frac{E}{p} = \frac{p}{2m} = \frac{v}{2} \tag{10.63}$$

[INSERT FOURIER ANALYSIS PICTURES] The standard deviation of the Fourier transforms of these Gaussians are,

$$\sigma_x \sigma_k = \frac{1}{2} \tag{10.64}$$

10.5.1 Heisenberg Uncertainty

- consider an electron with wave packet $\psi(x,t)$ then the most probable position is given by the maximum of $\mid \phi(x,t) \mid^2$.
- over a series of measurements over electrons with identical wave functions, we will measure, $\mid \phi(x,t) \mid^2$.
- If the distribution in position is narrow, the distribution in k must be wide,
- A narrow wave packet corresponds to small uncertainty in position and a large uncertainty in momentum

In general the ranges Δx and Δk are related by,

$$\Delta k \Delta x \sim 1 \tag{10.65}$$

Similarly, a wave packet that is localized in time must contain a wide range of frequencies,

$$\Delta\omega \Delta t \sim 1 \tag{10.66}$$

If we multiply these results by $\hbar$ then,

$$\Delta p \Delta x \sim \hbar \tag{10.67}$$
$$\Delta E \Delta p \sim \hbar \tag{10.68}$$

If the standard deviations are defined to be Δx and Δp, we know from the Fourier transform that the minimum value of their product is $1/2\hbar$. Now,

$$\Delta p \Delta x \geq \frac{\hbar}{2} \tag{10.69}$$
$$\Delta E \Delta p \geq \frac{\hbar}{2} \tag{10.70}$$

10.5.2 Schroedinger's Equation

How did Schrödinger arrive at his wave equation? Normally in a wave equation a second order time derivative is paired with a second order spacial derivative,

$$\frac{\partial^2 \Theta}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \Theta}{\partial t^2} \tag{10.71}$$

Schrödinger's equation involves only a first-order derivative in time. This was necessary in order to display de Broglie waves in the non-relativistic limit.

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + U(x)\psi = i\hbar\frac{\partial \psi}{\partial t} \tag{10.72}$$

Under certain circumstances we can write the wave function as the product of a time dependent function, and a time independent function,

$$\psi(x,t) = \psi(x)\exp[-i\omega t] \tag{10.73}$$

Now we can see if a function of this form is placed into the Schrödinger equation,

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + U(x)\psi &= i\hbar\frac{\partial \psi}{\partial t} && (10.74)\\ e^{-iwt}(-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x)}{\partial x^2} + U(x)\psi(x)) &= (-i\omega)(i\hbar)e^{-iwt} && (10.75)\\ -\frac{\hbar^2}{2m}\frac{d^2 \psi(x)}{dx^2} + U(x)\psi(x) &= E\psi(x) && (10.76) \end{aligned}$$

leads to a time independent Schrödinger equation, after the elimination of the time variable.

There are several important conditions to the wave equation,

- $\psi(x)$ must be continuous in x.
- $U(x)$ not infinite
- $\frac{d\psi}{dx}$ is continuous
- We also impose a normalization condition,

$$\int_{-infty}^{\infty} \mid \psi(x) \mid^2 dx = 1 \tag{10.77}$$

10.5.3 Particle in a one-dimensional box

A free quantum particle satisfies the Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t} \tag{10.78}$$

Let us try the solution,

$$\psi(x,t) = \phi(x)e^{-iE/\hbar} \tag{10.79}$$

Then,

$$\begin{aligned} e^{-iE/\hbar}(-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2}) &= (i\hbar)(-1\frac{E}{\hbar})e^{-iE/\hbar}\psi(x) && (10.80)\\ -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} &= E\phi(x) && (10.81)\\ -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} - E\phi(x) &= 0 && (10.82)\\ \frac{\partial^2\phi}{\partial x^2} + k^2\phi(x) &= 0 && (10.83)\end{aligned}$$

The solution of this equation is clearly,

$$\phi(x) = A\exp[\pm ikx] \tag{10.84}$$

[check it, if it satisfies the equation the uniqueness theorem says its true]

Now let us consider if we place an impenetrable box of dimension a between, $x = 0$ and $x = a$, this implies the boundary conditions of, $\phi(0) = \phi(a) = 0$.

The most general solution is now,

$$\phi(x) = A\exp[ikx] + B\exp[-ikx] \tag{10.85}$$

where $k = \sqrt{2mE/\hbar^2}$ and A, and B are constants, Applying the boundary condition $\phi(0) = 0$,

$$\phi(0) = A + B = 0 \tag{10.86}$$

There from A=-B, and we can write the solution as,

$$\phi(x) = C\sin kx \tag{10.87}$$

Where $C = 2iA$. Now we impose the boundary condition $\phi(a) = 0$

$$\sin ka = 0 \tag{10.88}$$

Therefore, $ka = n\pi$, This means that the wave number is quantized,

$$k_n = \frac{n\pi}{a} \tag{10.89}$$

and so is the energy,

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2} \tag{10.90}$$

with $n = 1, 2, 3\cdots$

This quantization is a result of the boundary conditions imposed on the Schrödinger equation. We now write the wave function as,

$$\phi_n = C \sin\frac{n\pi x}{a} \tag{10.91}$$

We normalize this,

$$\int_0^a \mid \phi_n \mid^2 dx = 1 \tag{10.92}$$

and we find that $C = \sqrt{2/a}$,

$$\phi_n = \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a} \tag{10.93}$$

The complete solution is given as,

$$\psi_n(x,t) = \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a} exp[-i\frac{E_n t}{\hbar}] \tag{10.94}$$

And the probability of finding a particle at position x at in a stationary state $\psi_n(x,t)$ is,

$$P_n(x,t) =\mid \psi_n(x,t) \mid^2 = \frac{2}{a}\sin^2\frac{n\pi x}{a} \tag{10.95}$$

Since the Schrödinger equation is linear, a superposition of states is also a solution,

$$\psi(x,t) = \sqrt{\frac{1}{2}}[\psi_{n'}(x,t) + \psi_{n''}(x,t)] \tag{10.96}$$

We calculate the the probability P(x,t) for such a wave,

$$P_n(x,t) =\mid \psi_n(x,t) \mid^2 = \frac{1}{a}\Big[\sin^2\frac{n'\pi x}{a} + \sin^2\frac{n''\pi x}{a} \tag{10.97}$$

$$+2\sin\frac{n'\pi x}{a}\sin n''\pi x a \cos[\frac{(E_{n'} - E_{n''})t}{\hbar}] \tag{10.98}$$

There is now oscillatory behaviour between this two states, the probability changes with time, and with a frequency of,

$$\nu = \frac{(E_{n'} - E_{n''})}{h} \tag{10.99}$$

which corresponds to the Bohr frequency.

Some additional points

1. the number n is know as a quantum number as it specifies which quantum state the system is in.

2. There is a certain minimum energy the system must have, $E_1 = \frac{\pi^2\hbar^2}{2ma^2}$

3. The distance between energy levels increases with decreasing a, as a gets larger we approach the classical limit of continuous energy.

10.5.4 Tunneling

- Some regions are forbidden by energy considerations
- $U > E_{TOT}$ particle would have to have a negative K
- Limited by potential barrier
- There is a small but finite probability of the particle penetrating the barrier: tunneling.

10.6 Spin

We all know that the world is not 1-dimensional, so we would like to write out (not solve) the 3-dimensional Schrödinger equation.

$$-\frac{\hbar^2}{2m}(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}) + U(x)\psi(x,t) = i\hbar\frac{\partial\psi}{\partial t} \tag{10.100}$$

If we combine this to a box of dimension $L \times L \times L$, then we can write the energies as,

$$E = \frac{\hbar^2}{2m}(k_1^2 + k_2^2 + k_3^2) \tag{10.101}$$
$$= \frac{\hbar^2\pi^2}{2mL^2}(n_1^2 + n_2^2 + n_3^2) \tag{10.102}$$

We can see there is a quantum number for each dimension.

Similarly we imagine that the confining potential of a [hydrogen] atom is not a cube, but rather a a spherically symmetric potential, thus we need to write the Schrödinger equation in spherical coordinates. The transformation need not concern us,

$$-\frac{\hbar^2}{2m}\Big[\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial\psi}{\partial}r\Big) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial\psi}{\partial\theta}\Big) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\Big] + U\psi = E\psi \tag{10.103}$$

Solutions for the spherical Schrödinger equation may be written as the product of functions for separate variables,

$$\psi(r,\theta,\psi) = R(r)f(\theta)g(\phi) \tag{10.104}$$

often however the two angular coordinates are coupled such that the solution becomes,

$$\psi_{nlm}(r,\theta,\psi) = R_{nl}(r)Y_{lm}(\theta,\phi) \tag{10.105}$$

Where $R_{nl}(r)$ are related to Laguerre polynomials, and $Y_{lm}(\theta,\phi)$ are spherical harmonics,

$$R_{nl}(r) = -\Big[\Big(\frac{2Z}{na_0}\Big)^3\frac{(n-l-1)!}{2n[(n+l)!]^3}\Big]^{1/2}\Big(\frac{2Zr}{na_0}\Big)^l e^{-Zr/na_0}P_{n+1}^{2l+1}\Big(\frac{2Zr}{na_0}\Big) \tag{10.106}$$

$$Y_{lm}(\theta,\phi) = \Big[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\Big]^{1/2}(-1)^m e^{im\phi}P_l^m(cos\theta) \tag{10.107}$$

$$P_l^m(\eta) = \frac{(-1)^m}{2^l}\frac{(l+m)!}{l!(l-m)!}(1-\eta^2)^{-m/2}\frac{d^{l-m}}{d\eta^{l-m}}(\eta^2-1)^l \tag{10.108}$$

Now I included here a lot of unimportant details, but what we notice is that there are still 3 quantum numbers, however now nlm are interdependent. The possible values of these quantum numbers are,

$$n = 1,2,3\cdots \tag{10.109}$$

$$l = 0,1,2,\cdots,n-1 \tag{10.110}$$

$$m = -l,-l+1,-l+2,\cdots,+l \tag{10.111}$$

n is known as the principle quantum number, l the orbital quantum number, m the magnetic quantum number.

When the spectral lines of hydrogen are viewed under high resolution, it is found to consist of two closely spaced lines. This splitting of a line is known as fine structure.

In 1925 W. Pauli suggested the electron has another quantum number that can take 2 values.

S. Goudsmit and G. Uhlenbeck (1925) suggest this quantum number is the $z-$component of the angular momentum of the electron with values, $m_s = 2s+1$, $s = \pm 1/2$.

10.7 The Copenhagen Interpretation

Chapter 11

Quantum Mechanics IV: Atomic Structure

When the S.E. was solved for hydrogen the results predicted the energy levels of the Bohr theory.

- Bohr model predicted one quantum number, n - principle quantum number
- SE supplied 2 other quantum numbers,
  - l - the orbital quantum number
  - m - the magnetic quantum number
- m shows up only when atoms are placed into a magnetic field. For a given, l, the level is split when placed into a magnetic field, into m_l levels.
- under high-resolution optical spectrometry each emission line is actually 2 lines.
- this splitting is known as fine structure. This spitting is characterized by the electron spin quantum number - m_s.
- Spin orientation is with respect to the atom's internal magnetic field, produced by orbit of an electron.

	Quantum number		number of allowed values
	n	$1, 2, 3, \ldots$	N/A
Quantum numbers	l	$0, 1, 2, \ldots, (n-1)$	n for each n
	m_l	$0, \pm 1, \pm 2, \pm 3, \ldots, \pm l$	$2l + 1$ for each l
	m_s	$\pm 1/2$	2

Other particles have spin.

11.1 Multi-electron Atoms

The Schroedinger Equation cannot be solved exactly for multi-electron atoms. We can however approximate multi-electron atoms with a hydrogen atom, the energies will depend on n, l.

- Electrons that share the same quantum number n make up a shell
- The sub-shell is defined by the quantum number l. It is common to use letter instead of numbers for the l quantum numbers,

s	p	d	f	g	...
0	1	2	3	4	...

an electron that is $2p$, $n = 2$ and $l = 1$, $3d$, $n = 3$, $l = 2$.

The numerical sequence of the filling of the shells depends on the energy of the shells, [INSERT PICTURE] we notice that 4s is below 3d.

How do electrons fill shells?

11.1.1 Pauli Exclusion Principle

Pauli Exclusion Principle: No two electrons in an atom can have the same set of quantum numbers.

Thus for $n = 1$ $l = 0$ (1s) there is only one $m_l = 0$ therefor there must be another unique quantum number $m_s = \pm 1/2$. There are only two electrons in this shell.

11.2 Periodic Table

The periodic table is a tool that is organized by quantum numbers. The vertical column are grouped to group elements of similar chemical properties. The electrical properties depend on the number of elections in the outer shell of the ground state. The rows (or periods) list the elements in increasing atomic number (number of protons)

We notice that the first period has 2 elements, periods 2-3, have 8 elements and period 4-5 have 18 elements.

Let us now think about quantum numbers, and how many electrons can have each quantum number.

- $n = 0$, $l = 0$, $m_l = 0$, $m_s = \pm 1/2$ so there are 2 electrons with $n = 0$
- $n = 1$, $l = 0$, $m_l = 0$, $m_s = \pm 1/2$ so there are 2 electrons with $n = 1$, $l = 0$; $l = 1$ $m = -1, 0, 1$ so there are 6 electrons with $n = 1$, $l = 1$
- $n = 2$ $l = 0, 1, 2$, there are 2 electrons with $n = 2, l = 0$, 6 with $n = 2, l = 1$ and 10 with $n = 2, l = 2$.

We can see that from this there should be 2 elements in the first period, 6 in the second and 18 in the third. Why the discrepancy, this is because the 4s level is lower in energy than the 3d level the 4s fills first and so the 3d elements are placed into the 4th period.

We can use the periodic table to express the quantum numbers of the ground state of an element. For example, Nickel, Ni, $1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^8$. The superscript indicated the number of electrons in the shell.

11.3 Heisenberg Uncertainty

In quantum mechanics on of the most important questions deals with the processes of measurement. Classically, physics is based upon determinism if we know the position and velocity of a particle at a particular time its future behavior can be predicted. Quantum theory is based on probability we stated earlier that $\mid \psi \mid^2$ give the probable position of the particle. Because of the probabilistic formulation there is a limit to the accuracy of any measurements.

In 1927 Werner Heisenberg, wrote a second approach to quantum mechanics that was later shown to complement the Schroedinger approach.

Heisenberg Uncertainty Principle: It is impossible to simultaneously know a particles exact position and momentum.

if we write the momentum as $p = \hbar k$.

- consider an electron with wave packet $\psi(x, t)$ then the most probable position is given by the maximum of $\mid \phi(x, t) \mid^2$.
- over a series of measurements over electrons with identical wave functions, we will measure, $\mid \phi(x, t) \mid^2$.
- If the distribution in position is narrow, the distribution k must be wide,
- A narrow wave packet corresponds to small uncertainty in position and a large uncertainty in momentum

In general the ranges Δx and Δk are related by,

$$\Delta k \Delta x \sim 1 \tag{11.1}$$

Similarly, a wave packet that is localized in time must contain a wide range of frequencies,

$$\Delta \omega \Delta t \sim 1 \tag{11.2}$$

If we multiply these results by $\hbar$ then,

$$\Delta p \Delta x \sim \hbar \tag{11.3}$$

$$\Delta E \Delta p \sim \hbar \tag{11.4}$$

If the standard deviations are defined to be Δx and Δp, we know from the Fourier transform that the minimum value of their product is $1/2\hbar$. Now,

$$\Delta p \Delta x \geq \frac{\hbar}{2} \tag{11.5}$$

$$\Delta E \Delta p \geq \frac{\hbar}{2} \tag{11.6}$$

A photon used to observer a particle changes its momentum.

11.4 Relativistic Quantum Mechanics

In 1928 P Dirac combined special relativity and quantum mechanics. One result was the prediction that for every charged particle the is an identical oppositely charged particle.

- positron charge $+e$ mass= m_e
- electron charge $-e$ mass= m_e

Positrons are created by pair production, electrons and positrons created together from the conservation of charge. x-ray passes nucleus creating +e and -e.

we can calculate the minimum energy needed from.

$$E_{min} = hf = 2m_e c^2 = 1.022MeV \tag{11.7}$$

Positrons captured when passing through matter join with an electron to create positronium this decays ($\approx 10^{-10}s$) into 2 photons.

11.5 Nuclear Structure

After the discovery of the electron the next question remaining was what is the structure of the atom. The common model now is based on the Rutherford-Bohr model, which is based upon the solar system. Most of the mass is contained in a small centrally located nucleus, with electrons orbiting around the nucleus like planets.

- Concept of a nucleus is based on experiments of the scattering of $\alpha-$particles.
- An alpha particle is a helium nucleus, these are naturally produced by some radio-active decay.
- $\alpha-$particles fired at a thin sheet of gold foil. Since the mass of the $\alpha-$particle is much greater than that of an electron, they will not scatter off of electrons,

$$m_\alpha > 7000 m_e \tag{11.8}$$

A simple estimate of size of a nucleus is given by its closest approach. We write the electrical potential as,

$$U = \frac{kq_1q_2}{r_{min}} = \frac{k(2e)(Ze)}{r_{min}} \tag{11.9}$$

The charge of a nucleus is $+Ze$ where Z is the number of protons.

Set this equal to the kinetic energy,

$$\frac{1}{2}mv^2 = \frac{k(2e)(Ze)}{r_{min}} \tag{11.10}$$

$$r_{min} = \frac{2kZe^2}{mv^2} \tag{11.11}$$

Rutherford found the radius of gold to be on the order of $10^{-14}m$

11.5.1 Nuclear Force

The forces between nucleons;

- Gravitational
- Electric force (between protons) Since the E. Force is ¿ than the gravitational force we need another attractive force to keep the nucleus stable.
- Nuclear force: strongly attractive ¿ than grav. and e.force, very short range. distance ($\sim 10^{-15}m$).
- Nuclear Force not related to charge.

11.6 Nuclear Notation

- The number of protons determines the species of the atom. This is usually expressed by a chemical symbol,

$$^{Z+N}_{Z}X_N \tag{11.12}$$

  Some times we only express the atomic mass (Z+N) since the chemical symbol gives Z. then we might write carbon as,

$$^{12}C \tag{11.13}$$

- Some atoms may contain a different numbers of neutrons, and the same number of protons these are known as isotopes. Isotopes of carbon are,

$$^{12}C,^{13}C,^{14}C,^{11}C,^{15}C,^{16}C \tag{11.14}$$

  Some Isotopes are more stable than others. The common isotope is the most stable. In carbon only $^{12}C,^{13}C$ are stable.

- Isotopes of hydrogen; ^1H, ^2H, ^3H.
- ^2H is known as deuterium (chemical symbol D) and forms heavy water D_2O.

11.7 The Laser

The laser is a result of theoretical exploration. LASER: Light Amplification by Stimulated Emission of Radiation Usually an electron makes a transition to a lower energy level almost immediately $10^{-8}s$. Some lifetimes can be appreciably longer. A state with a long life time is called "metastable" Materials with long metastable states "glow-in-the-dark" after the excitation is removed, this is called "Phosporescence".

11.7.1 Absorption and Spontaneous Emission

- Photon absorbed and emitted almost simultaneously
- If the higher state is metastable 3 possibilities,
  1. Absorption
  2. Emission
  3. Stimulated Emission
- Stimulated photon identical to the first photon
- The medium must be prepared, such that more atoms are in the metastable state then not. (poulation inversion)
- Mirrors places at the ends of the medium to enhance spontaneous emission
- Produces monochromatic/ coherent light
- because of coherence - very little dispersion.

11.7.2 Holography

[INSERT IMAGE]

Chapter 12

Quantum Mechanics V: Radiation

12.1 Radioactivity

Nuclei that are unstable break down

- Spontaneously discintegrate or decay
- they are termed radioactive
- Decay at a fixed-rate (do not discintegrate at the same time)
- Discovered by Henri Becquerel 1896 - Uranium (U)
- 1898 Pierre and Marie Curie, discovered Radium, Polonium

3 Types of Radiation:

- α $(+2e)$ Helium nucleus 2p+2n
- β $(-e)$ electron
- γ high energy quanta of electromagnetic energy (photon)

Because of the different charges the radiation can be distiquished by passing it through a magnetic field.

12.1.1 Alpha Decay

When an α-particle is ejected the mass number is reduced by 4 ($\Delta A = -4$ and $\Delta Z = -2$)

$$^{214}_{84}Po \rightarrow^{210}_{82} Pb +^{4}_{2} He \tag{12.1}$$

The decay of polonium to lead by α decay.

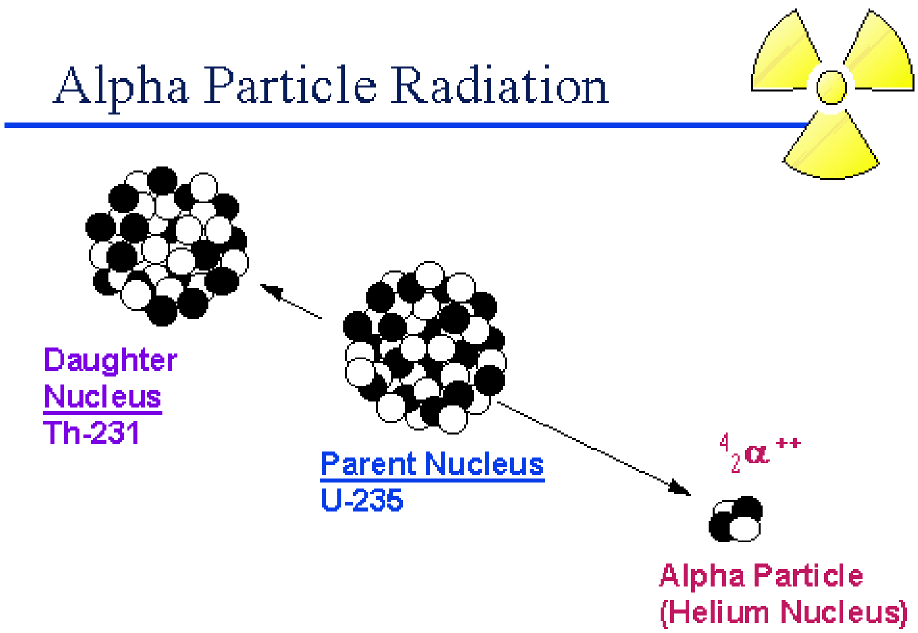
- conservation of nucleons
- conservation of charge

12.1.2 Example

U-238 decays by an alpha particle what is the daughter nucleus? A=238, Z=92 after the decay Z=90, A=234. The daughter nucleus is Thorium-234

$$ {}^{238}_{92}U \rightarrow {}^{234}_{90}Th + {}^{4}_{2}He \tag{12.2} $$
$$ {}^{238}_{92}U \rightarrow {}^{234}_{90}Th + \alpha \tag{12.3} $$

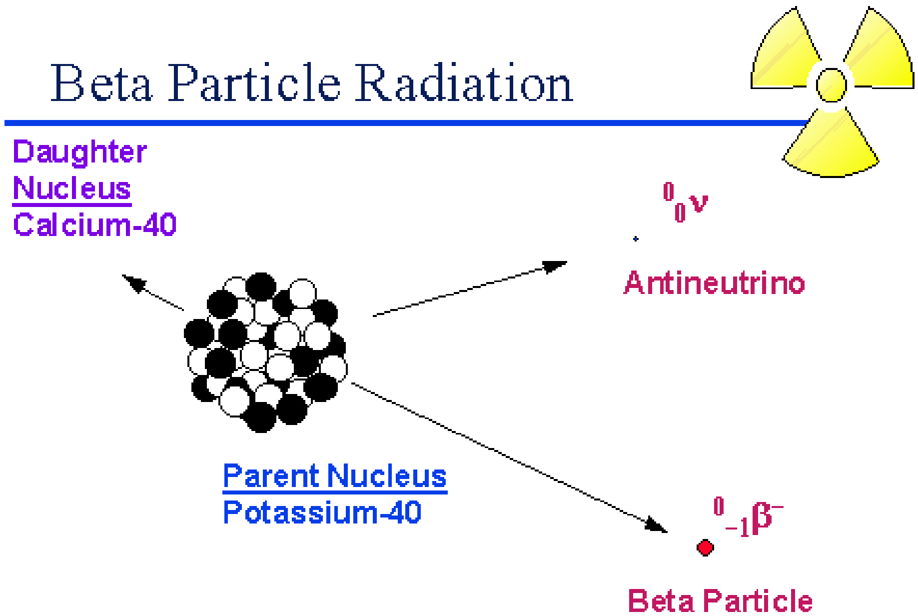
The kinetic energy of the alpha particle is typically a few MeV. $\begin{array}{lll} {}^{214}Po & \Rightarrow & 7.7MeV \\ {}^{238}U & \Rightarrow & 4.14MeV \end{array}$



Since the alpha particles have less energy than the barrier, classically they cannot pass. Quantum mechanically we see that the alpha in U-238 commonly crosses the barrier. The answer is due to tunneling. Tunneling is the name we give to the fact that there is a non-zero probability of an $\alpha-$particle initially on the outside being found inside of the barrier.

12.1.3 Beta Decay

- $\beta-$
  - An electron is emitted
  - Electron is created during the decay
  - the new negative electron is emitted $\rightarrow \beta-$



- A neutron decays to a proton and an electron
- occurs when too many neutrons as compared to protons
- neutrino also emitted

$$
\begin{aligned}
{}^{14}_{6}C &\rightarrow {}^{14}_{7}N + {}^{0}_{-1}e \qquad (12.4)\\
{}^{1}_{o}n &\rightarrow {}^{1}_{1}p + {}^{0}_{-1}e \qquad (12.5)
\end{aligned}
$$

- $\beta+$ positron decay
  - too many protons relative to neutrons
  - neutrino also emitted
  - the new positive electron is emitted $\rightarrow \beta+$
  - positron is emitted and neutron created.

$$
\begin{aligned}
{}^{15}_{8}O_7 &\rightarrow {}^{15}_{7}N_8 + {}^{0}_{1}e \qquad (12.6)\\
{}^{1}_{1}p &\rightarrow {}^{1}_{0}n + {}^{0}_{1}e \qquad (12.7)
\end{aligned}
$$

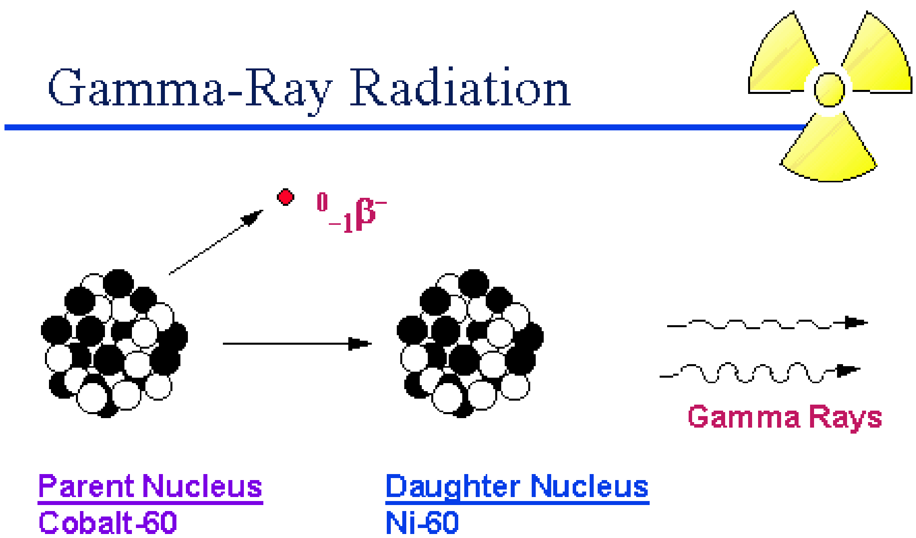
- Electron Capture
  - Absorbtion of orbital electron

– result similar to positron decay

– proton changes into a neutron.

$$ {}^{0}_{-1}e + {}^{7}_{4}Be \rightarrow {}^{7}_{3}Li \quad (12.8) $$
$$ {}^{1}_{1}p + {}^{0}_{-1}e \rightarrow {}^{1}_{0}n \quad (12.9) $$

12.1.4 Gamma Decay



- nucleus emits a γ ray (a high energy photon)
- often from an excited daughter nucleus from a α or β decay.
- nuclei have energy levels
- excited nuclei are marked by an asterix
- mass numbers do not change

$$ {}^{61}_{28}Ni^{*} \rightarrow {}^{61}_{28}Ni + \gamma \quad (12.10) $$

12.1.5 Example

the only stable isotope of cesium is Cs-133. Cs-137 is an unstable byproduct of nulcear power. Its decay often leads to an excited Barium nucleus, that emits a γ ray.

We notice that Cs-137 has too many neutrons (55p) so it decays by emission of a $\beta-$.

$$^{137}_{55}Cs \rightarrow {}^{137}_{56}Ba^* + \beta - \quad (12.11)$$

$$^{137}_{56}Ba^* \rightarrow {}^{137}_{56}Ba + \gamma \quad (12.12)$$

12.1.6 Radiation Penetration

The pentration of radioactive particles is important for shielding and other uses, such as the use of γ rays to sterilize food, and using absorption to control the thickness of metals.

- α-particles
  - Doubly charged, massive
  - very slow
  - stopped by a few centimeters of air or a sheet of paper
- β-particles
  - less massive
  - stopped by a few meters of air or milimeters of aluminium
- γ rays
  - can penetrate several centimeters of dense materials like lead Pb
  - Lead is commonly used as shielding for x-rays or gamma-rays
  - Removed/lose energy by compton scattering, photoelectric effect, pair production.
- Damage from radiation is from the ionization of living cells
- Continually exposed to normal background radiation
- Workers in nuclear industries are monitored
- only a small number of unstable nuclides occur naturally
- $U - 238 \rightarrow Pb - 206$ after many steps

12.2 Half-life

Nuclei decay randomly at a rate characteristic of the particular nucleus. We can determine how many decays in a sample for given time, but not when a particular nucleus will decay.

- activity (R) number of clear disintegrations per second.
- Activity decreases with time

$$\frac{\Delta N}{\Delta t} = -\lambda N \tag{12.13}$$

$$R = \mid \frac{\Delta N}{\Delta t} \mid = \lambda N \tag{12.14}$$

$$N = N_o e^{-lambdat} \tag{12.15}$$

where N is the number of atoms, N_o is the inital number, λ is the decay constant.

Decay rate is often expressed in half-life. or the time it takes for half of the sample to decay.

$$\frac{N}{N_o} = \frac{1}{2} \tag{12.16}$$

After one half-life the activity is cut in half.

12.2.1 Example: Sr-90

the half-life of $Sr - 90$ is 28yrs. If we have $100\mu g$ after 28yrs there is $50\mu g$ of Sr

$$^{90}_{38}Sr \rightarrow ^{90}_{39} Y + ^{0}_{-1} e \tag{12.17}$$

Half life of u-238 is 4.5 Billion years if the universe is 10 Billion years old about 1/2 the original amount is still around.

we can find the half life from,

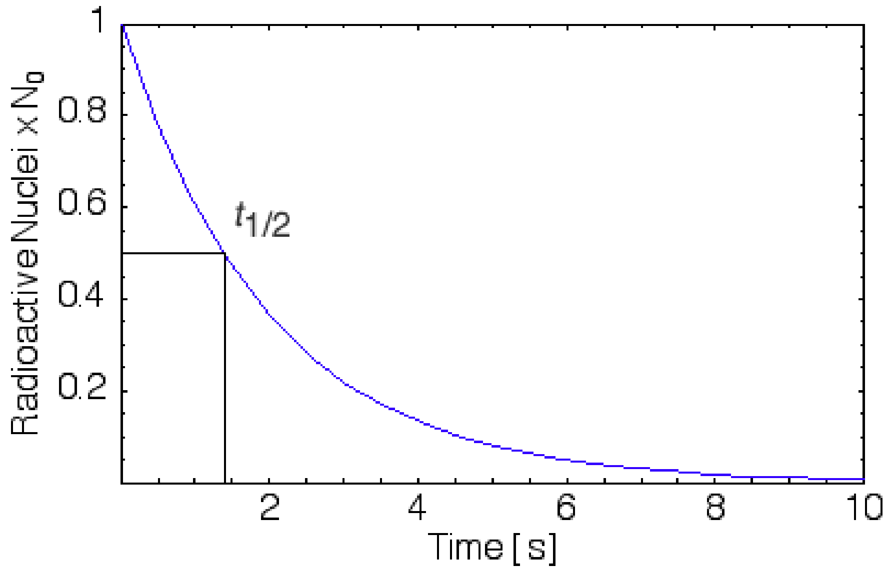
$$\frac{N}{N_o} = \frac{1}{2} = e^{-\lambda t} \tag{12.18}$$

$$\ln(\frac{1}{2}) = -\lambda t \tag{12.19}$$

$$t_{1/2} = \frac{0.693}{\lambda} \tag{12.20}$$

Units:

- $1Ci = 3.70 \times 10^{10} Decays/s$
- $1Decay/s = 1Bq$ (Bq=Bequeral)



12.2.2 Radioactive Dating

- Because decay rates are constant they can be used as clocks
- we can caculate backwards into time
- C-14: Living objects have a constant amount of C-14 equal to the amount in the atmosphere, if we assume the amount of carbon in the atmosphere has not change appreciably then we can calculate the age of something by the amount of carbon 14.
- current density 1 $C - 14$ per $7.2 \times 10^{11} C - 12$
- $t_{1/2} = 5730 yrs$
- C-14 is produced by cosmic rays

$$p+ \rightarrow n \tag{12.21}$$
$$^{14}_{7}N + n \rightarrow ^{14}_{6}C +^{1}_{1} H \tag{12.22}$$
$$^{14}_{6}C \rightarrow ^{14}_{7}N + \beta- \tag{12.23}$$

- intensity of cosmic rays are not constant
- when organism dies ^{14}C starts to decay
- Conentration of ^{14}C can be used to date the death.

Table 12.1: Nuclear Stability

Number of stable isotopes	Z	N
168	even	even
107	even(odd)	odd(even)
4	odd	odd

12.2.3 Example: Carbon 14 dating

If there are 20 decays/min in 10g of carbon or 2.0decay/(gm). If the half life in a living organism is 16 decays/gm. How long ago was the organism living?

$$16t \rightarrow^{t_{1/2}} 8 \rightarrow^{t_{1/2}} 4 \rightarrow^{t_{1/2}} 2 \tag{12.24}$$

$$3(t_{1/2}) = 3(5730yr) = 1.7 \times 10^4 y \tag{12.25}$$

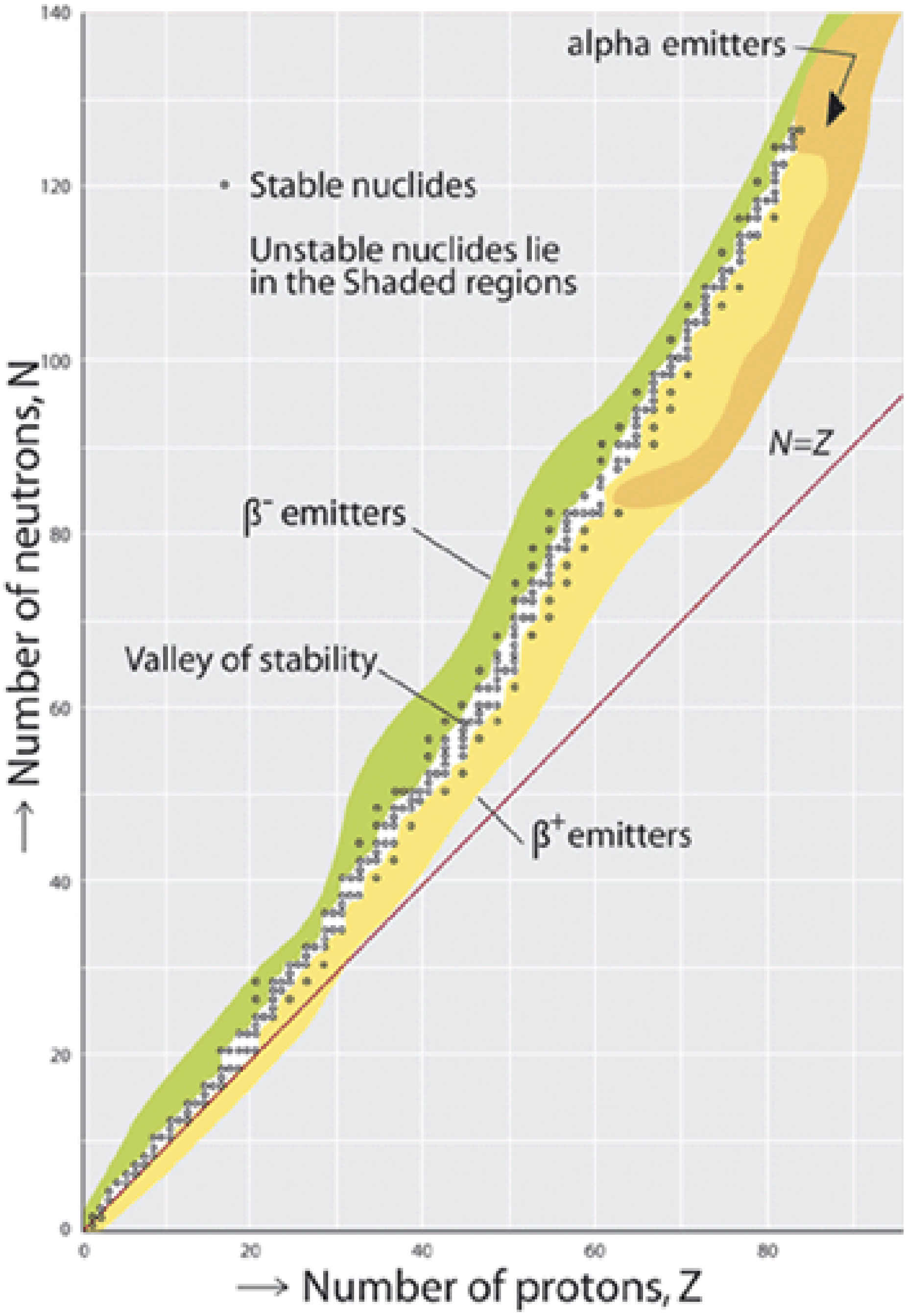
12.3 Nuclear Stability

- Depends on dommance of nuclear force over repulsive forces.
- $A < 40$ $N/Z \approx 1$
- $A > 40$ $N/Z > 1$
- Radioactive decay occurs until the nucleus lands on the stability curve

Often times there is pairing between similar nucleons, look at the number of stable isotopes and the number of protons and neutrons,

Criterial for stability;

1. $Z > 83$ unstable
2. Pairing
   (a) most e-e are stable
   (b) many o-e or e-o are stable
   (c) only 4 o-o are stable ${}^2_1H$, ${}^6_3Li$, ${}^{10}_4Be$, ${}^{14}_7N$
3. number of nucleons
   (a) $A < 40$ $Z \approx N$
   (b) $A > 40$ $N > Z$


alpha emitters
Stable nuclides
Unstable nuclides lie in the Shaded regions
β⁻ emitters
Valley of stability
β⁺emitters
N=Z
⟶ Number of neutrons, N
⟶ Number of protons, Z
0
20
40
60
80
100
120
140

12.3.1 Binding Energy

- can be calculated by the mass-energy equivalence
- atomic mass unit (u) $1u = 1.66 \times 10^{-27} kg$
- E of one mass unit,

$$
\begin{aligned}
E &= mc^2 && (12.26) \\
&= (1.66 \times 10^{-27} kg)(2.9977 \times 10^8 m/s^2) = 1.4922 \times 10^{-10} J && (12.27) \\
&= 931.5 MeV && (12.28)
\end{aligned}
$$

Let us calculate the binding energy in terms of hte energy of hydrogen. Let us calculate the mass of He and compair it to the mass of 2 hydrogen.

- mass of 2 H= 2.01565u
- mass of 2 n = 2.07330u
- sum of 2H + 2n=4.03980u
- mass of ^4He =4.002603u
- the mass difference $\Delta m = 4.03980u - 4.002603u = 0.030377u$
- Converting the mass difference to energy gives the binding energy, $E = 28.30 MeV = \Delta mc^2$
- Average per nucleon,

$$\frac{E_B}{A} = \frac{28.30 MeV}{4} = 7.075 MeV/nucleon \quad (12.29)$$

- Much stronger than the binding energy of electrons.

- If a massive nucleus is split (fissioned) the nucleons on the daughter nuclei are more tightly bound and energy is released
- if 2 light nuclei are fused (fusion) the daugther nuclei is more tightly bound and energy is released
- $E_B \propto 1/A$ except for light nuclei
- This implies nearest neighbor interactions

There are magic numbers, if Z=2,8,20, 28, 50, 82, 126, if an element has a magic number of Z many isotopes are stable.

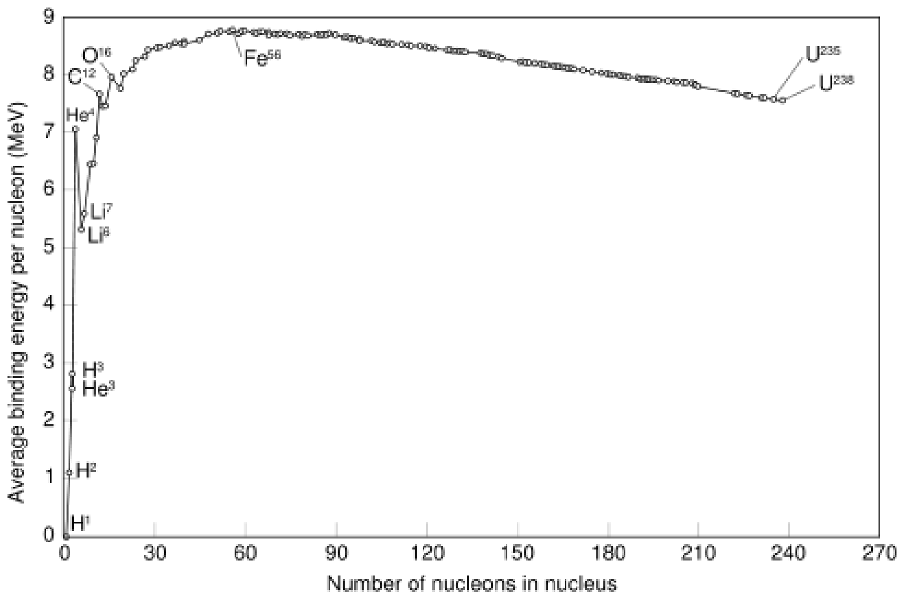


Figure 12.1: Nuclear Binding Energy

12.4 Radiation Detection

We can detect radiation by,

- α- β transfer energy by electrical interactions.
- γ compton scattering, photo-electric effect
- Particles produced by the interactions above can be detected.

12.4.1 Gieger Counter

- Hans Gieger (1882-1945) Student of Ernest Rutherford
- Ionization of a gas creates a current event which is amplified and detected

12.4.2 Scinillation Counter

Atoms of phosphor material (NaI) is excited the excitation is measured by a photo-electric material (Photo-multiplier tube)

12.4.3 Semi-conductor Detector

Charged particles stike a semiconductor producing electrons which can be measured

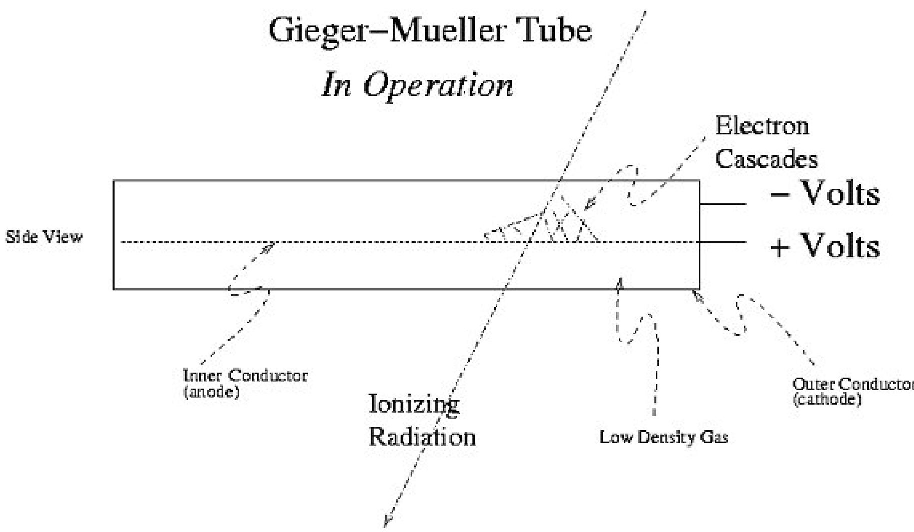


Figure 12.2: Geiger-Meuller tube

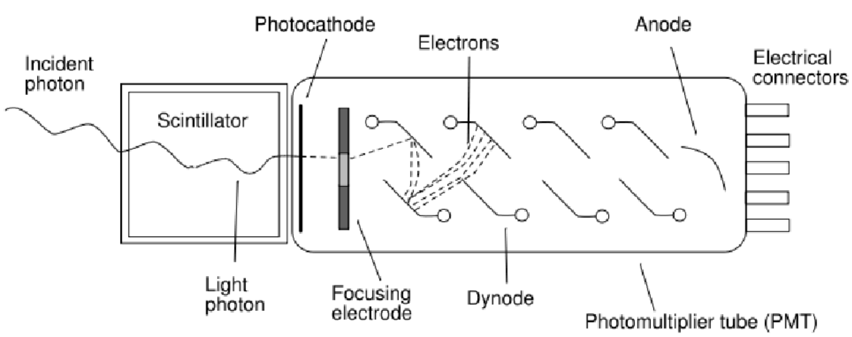


Figure 12.3: Photomultiplier tube

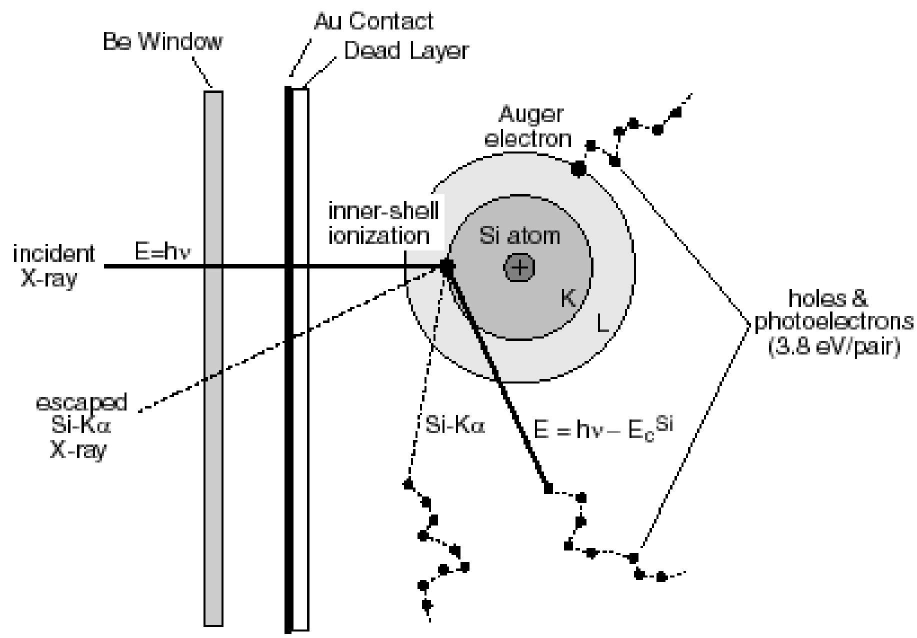


Figure 12.4: Si-Li Detector

12.4.4 Cloud Chamber

- Shows trajectory of the particle
- cloud or bubble chamber created when a vapor or a liquid is super cooled by varying the volume and the pressure
- spark super heated vapor created by varying the volume and the pressure

12.5 Biological Effects and Medical Applications

- Radiation Dosage:
  - Rotegen (R): number of x-rays or γ-rays required to produce an ionization charge of $2.58 \times 10^{-4} C/kg$
  - rad -1 rad - absorbed dose of $10^{-2} J/kg$
  - Gray (Gy) - Si Unit 1 Gy=100rad
- ionization of water on a cell can damage or kill a cell.
- Effective dose : Depends on the type of radiation

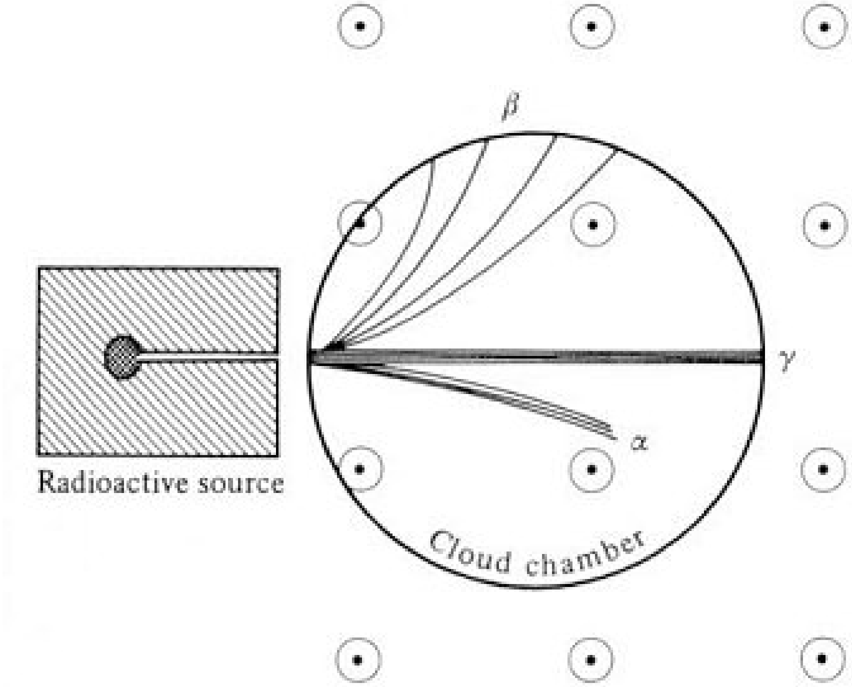


Figure 12.5: Cloud Chamber schematic

Figure 12.6: Ions in a cloud chamber

- rem (rad equivalent man)
- RBE relative biological effectiveness
- RBE for x-ray, gamma-ray RBE=1
- $rem = rad \times RBE$
- 20 rem α (1rad) same damage as 20 rem x-ray (20rad)
- SI unit (Sv)

12.6 Nuclear Reactions

Nuclear reactions occur naturally when a nuclei decays into new nuclei.

The first artificially produced nuclear reaction was by Rutherford, who bombared neutrons with alpha particles to produce a proton.

One example of a nuclear reaction,

$$^{14}_{7}N +^{4}_{2} He \rightarrow^{17}_{8} O +^{1}_{1} H \tag{12.30}$$

The general form of a reaction is

$$A + a \rightarrow B + b \tag{12.31}$$

or $A(a,b)B$

Atoms that are not stable can be created through nuclear reactions ($Z > 83$)

12.6.1 Conservation of mass-energy

In every nuclear reaction, the total relativistic energy must be conserved.

$$(K_N + m_N c^2) + (K_\alpha + m_\alpha c^2) = (K_O + m_O c^2) + (K_p + m_N p^2) \tag{12.32}$$

$$K_O + K_p - K_N - K_\alpha = (m_N + m_\alpha - m_O - m_p)c^2 \tag{12.33}$$

The Q value is defined by the change in kinetic energy,

$$Q = \Delta K = (K_O + K_p) - (K_N + K_\alpha) \tag{12.34}$$

$$= (m_N + m_\alpha - m_O - m_p)c^2 \tag{12.35}$$

Thus for a reaction we can have mass changing into kinetic energy and vice versa.

- if mass increases $K \downarrow$
- if mass decreases $K \uparrow$

If $Q < 0$ the reaction requires a minimum amount of energy to occur. The Q for $^{14}N(\alpha,p)^{17}O$.

$$\begin{aligned} Q &= (m_N + m_\alpha - m_O - m_p)c^2 && (12.36)\\ &= (14.003074u + 4.002603u - 16.99913u - 1.07825u)c^2 && (12.37)\\ &= (-0.001281u)c^2 && (12.38)\\ &= -1.193MeV && (12.39)\end{aligned}$$

- Endoergic $Q < 0$, $K \rightarrow m$
- Exoergic $Q > 0$, $m \rightarrow K$

Radioactive decay $Q > 0$ (always) if $Q < 0$ the minium kinetic energy required is,

$$K_{min} = (1 + \frac{M_a}{M_A}) \mid Q \mid \qquad (12.40)$$

the minimum K is greater than Q because of the conservation of linear momentum.

12.6.2 Example

When a K_{min} greater than the values for several possible reactions the result is governed by the probability from Quantum Mechanics. The reaction's cross-section is a measure of the probability.

12.7 Nuclear Fission

In early attempts to make heavy nuclei, neutrons were shot at uranium atoms. sometimes when a neutron strikes a U-238 molecule the nucleus will split forming nuclei of lighter elements. This is known as nuclear fission.

In a fission reaction, a heavy nucleus splits into two smaller nuclei plus some energy. Spontaneous fission is a slow process. Fission can be induced. Induced fission is the basis of nuclear power and nuclear bombs.

$$\begin{aligned} {}^{235}_{92}U +^1_0 n \rightarrow ({}^{236}_{92}U) &\rightarrow {}^{140}_{54}Xe +^{94}_{38} Sr + 2({}^1_0n) && (12.41)\\ &\rightarrow {}^{141}_{56}Ba +^{92}_{36} Kr + 3({}^1_0n) && (12.42)\\ &\rightarrow {}^{150}_{60}Nd +^{81}_{32} Ge + 5({}^1_0n) && (12.43)\end{aligned}$$

- Only certain nuclei can undergo fission
- Probability depends on the speed of neutrons
  - Slow neutrons $K \approx 1eV$ ^{235}U, ^{239}Pu

  - Fast neutrons $K > 1MeV$ ^{232}Th

- E_b/A curves give the estimate of the energy released
  - $^{235}U \rightarrow$
- Much more energy released than burning coal
- Chain reaction : the neutrons released cause future fissions
- the number of neutrons doubles w/ each generation, energy growth is expoential
- To maintain a chain reaction you must have a minimum amount of fissible material
  - Crtical mass
  - At least one nuetron creates a new event
- ^{235}U, ^{238}U
  - ^{238}U can absorb neutrons and not fission
  - ^{235}U about 0.7% of naturally occuring uranium
  - Enriching process concentrates ^{235}U to reduce the critical mass
    * 3% to 5% reactor grade
    * 99% Weapons grade
- If a chain reaction is not controlled, we have an explosion
  - A nuclear bomb maybe created by joining several sub-critical masses into a super-critical mass.

12.7.1 Nuclear Power

Fuel rods of enriched uranium are seperated by control rods (boron-cadmeium) and immersed in water. The reaction heats the water which intern is used to heat uncontaminated water to steam which turns a turbine just like a coal plant.

The control rods remove neutrons to slow the reaction. Fully inserted they stop the reaction.

The core must always be cooled because of spontaneous decay. Water also acts as a moderator the speed of the emitted neutrons are slowed by the water, often this enables the chain reaction.

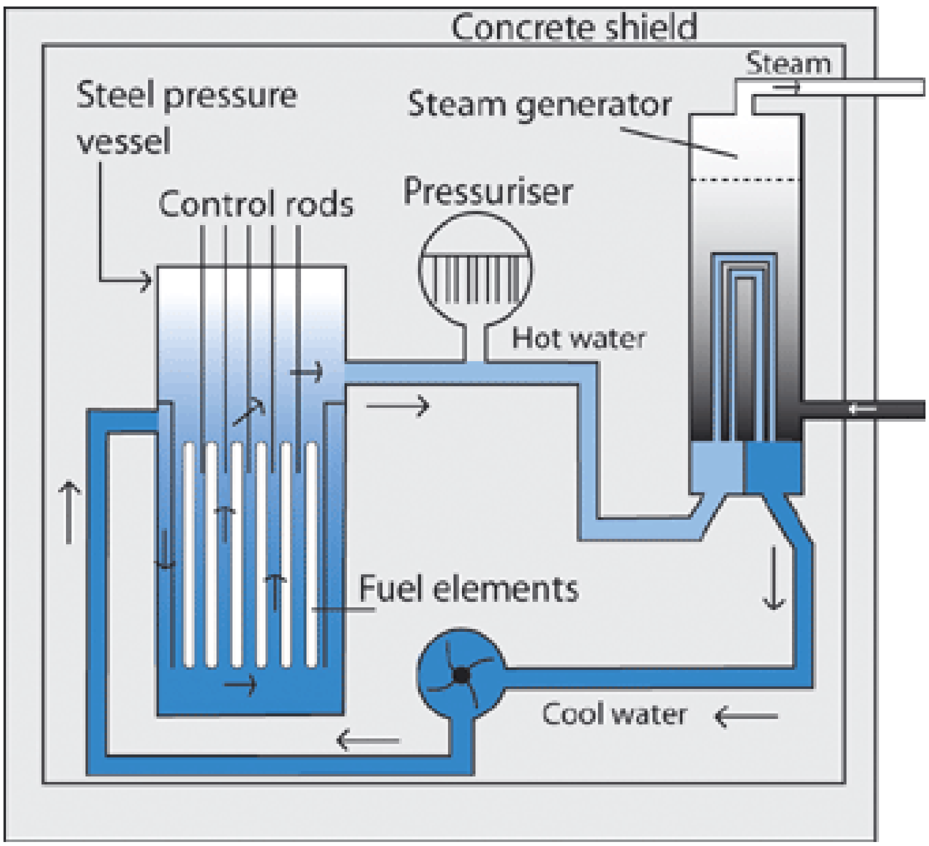


Figure 12.7: Nuclear Reactor

12.7.2 Breeder Reactor

^{238}U + fast neutrons

$$^1_0n +^{238} U \rightarrow^{238}_{92} U^* \rightarrow^{239}_{93} Np + \beta- \rightarrow^{239}_{94} Pu + \beta- \tag{12.44}$$

$^{239}_{94}Pu$ is a fissionable product. A breader reactor uses this reaction to generate plutonium for nuclear reactors (or bombs). Development in the US stopeed in the 1970's but it is big in france.

12.7.3 Safety

- LOCA : loss of coolant accident
  - coolant fails
  - rods heat-melt-fracture
  - fissing mass fall to floor into coolant causing steam or hydrogen explosion
- meltdown -melt can fall to the floor and melt through the floor
- TMI partial release of radioactive steam
- Chernobyl Meltdown - LOCA caused by human error and bad design and a huge explosion
- Where is waste stored

12.8 Nuclear Fusion

A fusion reaction is one where 2 light nuclei form a heavier nucleus.

$$^2_1H +^3_1 H \rightarrow {}^4_2He +^1_0 n + 17.6MeV \tag{12.45}$$

$$(2D)(D-D) \rightarrow 3.27MeV \tag{12.46}$$

A fusion reaction will release less energy per-reaction than a fusion reaction, but more energy per kg. The sun is powered by a fusion reaction.

$$^1_1H +^1_1 H \rightarrow^2_1 H + \beta^+\nu \tag{12.47}$$

The nuetrino is a subatomic particle generated by fusion reactions.

$$^1_1H +^2_1 H \rightarrow {}^3_2He + \gamma \tag{12.48}$$

$$^3_2He +^3_2 He \rightarrow {}^4_2He +^1_1 H +^1_1 H \tag{12.49}$$

If we summarize these reactions,

$$4(^1_1H) \rightarrow^4_2 He + \beta^+ + 2\gamma + 2\nu + Q \tag{12.50}$$

Q=+24.7MeV

12.8.1 Example: Fusion and the sun

given: $I = 1.40 \times 10^3 W/m^2$ striking the earth. what is the mass loss per second?
given: $R_{E-S} = 1.50 \times 10^8 km = 1.5 \times 10^{11} m$
$M_S = 2.00 \times 10^{30} kg$

$$\begin{aligned} A &= 4\pi R_{E-S}^2 = 2.83 \times 10^2 3m^2 && (12.51) \\ P_s &= IA = 3.96 \times 10^{26} J/s = 2.48 \times 10^{39} MeV/s && (12.52) \\ \frac{\Delta m}{\Delta t} &= \frac{(2.48 \times 10^{39} MeV/s)(1.66 \times 10^{-27} kg/u)}{931.5 Mev/u} && (12.53) \end{aligned}$$

12.8.2 Fusion Power

- Need D_2O whiclh occurs naturally in the oceans.
- less release of radioactive material than a fission reaction.
- Shorter half-life ^3H $t_{1/2} = 12.3 yrs$
- Technically no method yet to control
- Hydrogen bomb
- Gas of positively charged ions = plasma
- How to confine plasma in a specific density
  - Magnetic confinement
    - E-Field → current → heat (Tokomak)
    - minimum density and confinement time
  - Inertial Confinement
    - Pulses of laser explode H-pellets
    - Currently lasers are too week

12.9 Neutrino

The neutrinos introduced in order to conserve energy and linear momentum.

- Usually some missing energy
- Another particle is need, the neutrino

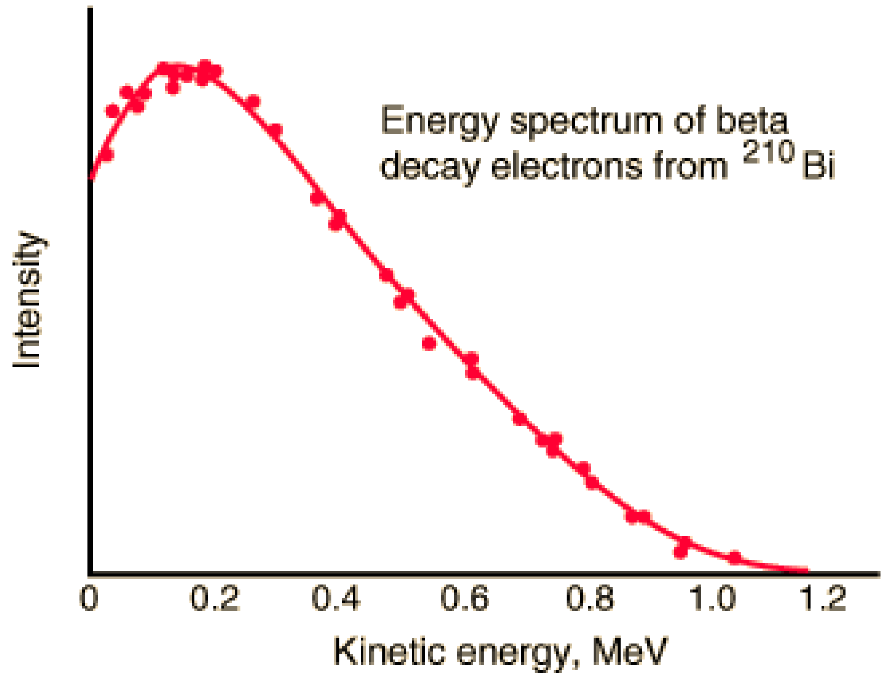


Figure 12.8: Maximum Kinetic energy

The neutrino interacts through the weak nuclear force.

$$ {}^{14}_{6}C \rightarrow {}^{14}_{7}N + \beta^{-} + \bar{\nu}_e \qquad (12.54)$$

$$ {}^{13}_{7}N \rightarrow {}^{13}_{6}C + \beta^{+} + \nu_e \qquad (12.55)$$

Chapter 13

Quantum Mechanics VI: The Standard Model

13.1 Beta Decay and neutrino

13.2 Fundamental Forces

The Heisenberg uncertainty principle $\Delta E \propto \frac{1}{\Delta t}$ allows for short term violation of energy conservation. or the reaction happens so fast that it is not observed. This is carried by a virtual or non-observed exchange particle.

Know forces in order of decreasing strength

1. Strong Nuclear Force
2. Electro-magnetic Force
3. Weak nuclear force
4. gravitational force

Particles that interact via strong

- Hadrons - make of quarks
  - Baryons half integer spin -3 quarks
  - mesons integer spin - quark + anti-quark

Particles that interact via week

- Leptons
  - electron
  - muons
  - neutrinos

13.3 Electro-Magnetic force and the Photon

The electro-magnetic forces interact via a virtual particle.

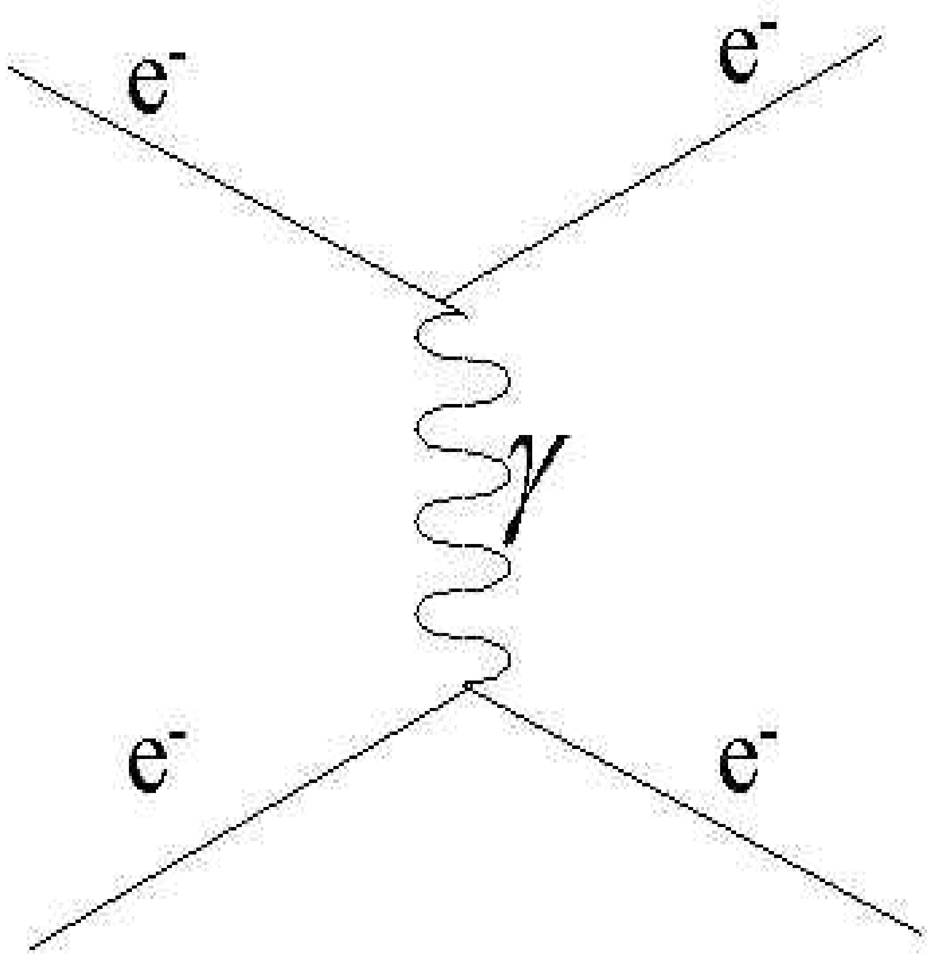


Figure 13.1: Feynman diagram of photon exchange

Change in momentum and energy of the particles is due to the virtual photon transfer.

13.4 Strong Force and mesons

In 1935 Hideki Yukawa (1907-1981) theorized a short range strong Nuclear force that held together the nucleus. The exchange particle that mediates the strong force is known as the meson.

The meson is a violation of the conservation of energy, by an amount of $\Delta E = m_m c^2$, this means that meson has to be absorbed in a time,

$$\Delta t \approx \frac{h}{2\pi \Delta E} R < c\Delta t = \frac{h}{2\pi m_m c} m_m \approx 270 m_e \tag{13.1}$$

Real mesons can be created by the collision of nucleon-nucleon interaction. Mesons were first observed in connection with cosmic rays in 1936. μ mu meson or a muon.

In 1947, the pion π meson was discovered.

$$\pi \to \mu^+ + \nu_\mu \tag{13.2}$$

ν_μ muon neutrino.

13.5 Weak nuclear force - W particle

Enrico Fermi discovered the W particle, that is created during nuclear decay. Free neutrons typically decay in 10.4min.

$$n \to p^+ + \beta^- + \bar{\nu}_e \tag{13.3}$$

The week nuclear force is very short range $R \sim 10^{-17} m$, since the range is very small the carrier must have a large mass.

The first observance of a non-virtual W occured in the 1980's The weak force is,

- weak is the only force on neutrinos
- transmitting identities of particles in the nucleus.
- Supernovas

13.6 Gravity

The mediating particle of the gravitational force is said to be the graviton.

Experiment $15 - 200 GeV$ electrons, muons, and neutrinos colliding with nucleons.

Table 13.1: Fundamental Forces

Force		Range	exchange	Particles
Fundamental Strong	1	$\approx 10^{-15} m$	π	quarks/gluons
E-M	10^{-3}	$1/r^2$ (∞)	photon	electrically charged
Weak	10^{-8}	$10^{-17} m$	W, Z	quarks/leptons
Gravity	10^{45}	$1/r^2$ (∞)	Graviton	all
Residual Strong			Mesons	Hadrons

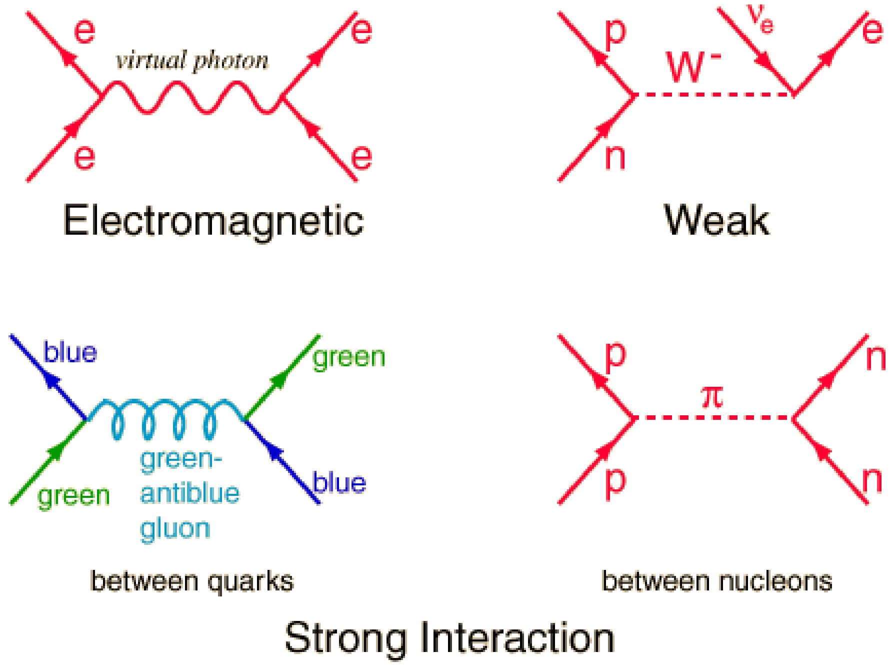


Figure 13.2: Feynman diagrams for the fundamental forces

Table 13.2: Quarks

up (+2/3e)	top (+2/3e)	strange (-1/3e)
down (-1/3e)	bottom (-1/3e)	charm (+2/3e)

13.7 Fundamental particles

- Particles made of other particles, is there a limit?
- Hadrons - particles that interact through the strong nuclear force. p, n, π.
- Leptons - interact through the weak force but not the strong force.

e	considered a point particle	
μ		
τ tauon		
ν neutrino	very small mass.	ν_e, ν_μ, ν_τ

- 1963 Gell-man, Zweig suggests that there are particles that make up hadrons. These particles are named quarks

Fundamental particles of the Hadron Family.
Current Elementary particles, Quarks and Leptons.
Free quarks have not been observed. QCD, quarks given color to satisfy pauli exclusion. Force between colors gluons.

13.8 Unification Theories

- Relativity unified gravitational, electro-magnetic and mechanics
- 1960's Electro-weak force unified the electro-magnetic and weak forces - Nobel Prize 1979 Glashow, salazar, Weinberg.
- GUT - Grand Unified Theory (not finished) to unify electro-weak and the strong force.

The standard model is the electoweak force and quantum chromodynamics.

Part IV

Applications of Quantum Mechanics

Chapter 14

Quantum Statistics

Chapter 15

Semiconductors

Appendices

Appendix A

Nobel Prizes in Physics

- 2008 - Yoichiro Nambu, Makoto Kobayashi, Toshihide Maskawa
- 2007 - Albert Fert, Peter Grnberg
- 2006 - John C. Mather, George F. Smoot
- 2005 - Roy J. Glauber, John L. Hall, Theodor W. Hnsch
- 2004 - David J. Gross, H. David Politzer, Frank Wilczek
- 2003 - Alexei A. Abrikosov, Vitaly L. Ginzburg, Anthony J. Leggett
- 2002 - Raymond Davis Jr., Masatoshi Koshiba, Riccardo Giacconi
- 2001 - Eric A. Cornell, Wolfgang Ketterle, Carl E. Wieman
- 2000 - Zhores I. Alferov, Herbert Kroemer, Jack S. Kilby
- 1999 - Gerardus 't Hooft, Martinus J.G. Veltman
- 1998 - Robert B. Laughlin, Horst L. Strmer, Daniel C. Tsui
- 1997 - Steven Chu, Claude Cohen-Tannoudji, William D. Phillips
- 1996 - David M. Lee, Douglas D. Osheroff, Robert C. Richardson
- 1995 - Martin L. Perl, Frederick Reines
- 1994 - Bertram N. Brockhouse, Clifford G. Shull
- 1993 - Russell A. Hulse, Joseph H. Taylor Jr.
- 1992 - Georges Charpak
- 1991 - Pierre-Gilles de Gennes

- 1990 - Jerome I. Friedman, Henry W. Kendall, Richard E. Taylor
- 1989 - Norman F. Ramsey, Hans G. Dehmelt, Wolfgang Paul
- 1988 - Leon M. Lederman, Melvin Schwartz, Jack Steinberger
- 1987 - J. Georg Bednorz, K. Alex Mller
- 1986 - Ernst Ruska, Gerd Binnig, Heinrich Rohrer
- 1985 - Klaus von Klitzing
- 1984 - Carlo Rubbia, Simon van der Meer
- 1983 - Subramanyan Chandrasekhar, William A. Fowler
- 1982 - Kenneth G. Wilson
- 1981 - Nicolaas Bloembergen, Arthur L. Schawlow, Kai M. Siegbahn
- 1980 - James Cronin, Val Fitch
- 1979 - Sheldon Glashow, Abdus Salam, Steven Weinberg
- 1978 - Pyotr Kapitsa, Arno Penzias, Robert Woodrow Wilson
- 1977 - Philip W. Anderson, Sir Nevill F. Mott, John H. van Vleck
- 1976 - Burton Richter, Samuel C.C. Ting
- 1975 - Aage N. Bohr, Ben R. Mottelson, James Rainwater
- 1974 - Martin Ryle, Antony Hewish
- 1973 - Leo Esaki, Ivar Giaever, Brian D. Josephson
- 1972 - John Bardeen, Leon N. Cooper, Robert Schrieffer
- 1971 - Dennis Gabor
- 1970 - Hannes Alfvn, Louis Nel
- 1969 - Murray Gell-Mann
- 1968 - Luis Alvarez
- 1967 - Hans Bethe
- 1966 - Alfred Kastler

- 1965 - Sin-Itiro Tomonaga, Julian Schwinger, Richard P. Feynman
- 1964 - Charles H. Townes, Nicolay G. Basov, Aleksandr M. Prokhorov
- 1963 - Eugene Wigner, Maria Goeppert-Mayer, J. Hans D. Jensen
- 1962 - Lev Landau
- 1961 - Robert Hofstadter, Rudolf Mssbauer
- 1960 - Donald A. Glaser
- 1959 - Emilio Segr, Owen Chamberlain
- 1958 - Pavel A. Cherenkov, Ilja M. Frank, Igor Y. Tamm
- 1957 - Chen Ning Yang, Tsung-Dao Lee
- 1956 - William B. Shockley, John Bardeen, Walter H. Brattain
- 1955 - Willis E. Lamb, Polykarp Kusch
- 1954 - Max Born, Walther Bothe
- 1953 - Frits Zernike
- 1952 - Felix Bloch, E. M. Purcell
- 1951 - John Cockcroft, Ernest T.S. Walton
- 1950 - Cecil Powell
- 1949 - Hideki Yukawa
- 1948 - Patrick M.S. Blackett
- 1947 - Edward V. Appleton
- 1946 - Percy W. Bridgman
- 1945 - Wolfgang Pauli
- 1944 - Isidor Isaac Rabi
- 1943 - Otto Stern
- 1942 - The prize money was with 1/3 allocated to the Main Fund and with 2/3 to the Special Fund of this prize section

- 1941 - The prize money was with 1/3 allocated to the Main Fund and with 2/3 to the Special Fund of this prize section
- 1940 - The prize money was with 1/3 allocated to the Main Fund and with 2/3 to the Special Fund of this prize section
- 1939 - Ernest Lawrence
- 1938 - Enrico Fermi
- 1937 - Clinton Davisson, George Paget Thomson
- 1936 - Victor F. Hess, Carl D. Anderson
- 1935 - James Chadwick
- 1934 - The prize money was with 1/3 allocated to the Main Fund and with 2/3 to the Special Fund of this prize section
- 1933 - Erwin Schrdinger, Paul A.M. Dirac
- 1932 - Werner Heisenberg
- 1931 - The prize money was allocated to the Special Fund of this prize section
- 1930 - Sir Venkata Raman
- 1929 - Louis de Broglie
- 1928 - Owen Willans Richardson
- 1927 - Arthur H. Compton, C.T.R. Wilson
- 1926 - Jean Baptiste Perrin
- 1925 - James Franck, Gustav Hertz
- 1924 - Manne Siegbahn
- 1923 - Robert A. Millikan
- 1922 - Niels Bohr
- 1921 - Albert Einstein
- 1920 - Charles Edouard Guillaume
- 1919 - Johannes Stark
- 1918 - Max Planck

- 1917 - Charles Glover Barkla
- 1916 - The prize money was allocated to the Special Fund of this prize section
- 1915 - William Bragg, Lawrence Bragg
- 1914 - Max von Laue
- 1913 - Heike Kamerlingh Onnes
- 1912 - Gustaf Daln
- 1911 - Wilhelm Wien
- 1910 - Johannes Diderik van der Waals
- 1909 - Guglielmo Marconi, Ferdinand Braun
- 1908 - Gabriel Lippmann
- 1907 - Albert A. Michelson
- 1906 - J.J. Thomson
- 1905 - Philipp Lenard
- 1904 - Lord Rayleigh
- 1903 - Henri Becquerel, Pierre Curie, Marie Curie
- 1902 - Hendrik A. Lorentz, Pieter Zeeman
- 1901 - Wilhelm Conrad Rntgen

Appendix B

Maxwell's Equations in the integral form

$$
\begin{aligned}
\oint_s E_n dA &= \frac{1}{\epsilon_0} Q_{inside} && \text{(B.1)}\\
\oint_s B_n dA &= 0 && \text{(B.2)}\\
\oint_c E\dot{d}l &= -\frac{d}{dt}\int_s B_n dA && \text{(B.3)}\\
\oint_c B\dot{d}l &= \mu_0 I + \mu_0\epsilon_0 \frac{d}{dt}\int_s E_n dA && \text{(B.4)}
\end{aligned}
$$

B.0.1 Maxwell's Equations in the derivative form

$$
\begin{aligned}
\nabla \cdot E &= \frac{1}{\epsilon_o}\rho && \text{(B.5)}\\
\nabla \cdot B &= 0 && \text{(B.6)}\\
\nabla \times E &= = -\frac{\partial B}{\partial t} && \text{(B.7)}\\
\nabla \times B &= \mu_o J + \mu_o \epsilon_o \frac{\partial E}{\partial t} && \text{(B.8)}
\end{aligned}
$$

in regions where there is no charge or current, we write Maxwell's equations as,

$$
\begin{aligned}
\nabla \cdot E &= 0 && \text{(B.9)}\\
\nabla \cdot B &= 0 && \text{(B.10)}\\
\nabla \times E &= = -\frac{\partial B}{\partial t} && \text{(B.11)}\\
\nabla \times B &= \mu_o \epsilon_o \frac{\partial E}{\partial t} && \text{(B.12)}
\end{aligned}
$$

Now we apply the operator $\nabla\times$ to the second two equations to get,

$$\nabla\times(\nabla\times E)=\nabla(\nabla\cdot E)-\nabla^2 E \;=\; \nabla\times(\frac{\partial B}{\partial t}) \tag{B.13}$$

$$= -\frac{\partial}{\partial t}(\nabla\times B)=-\mu_o\epsilon_o\frac{\partial^2 E}{\partial^2 t} \tag{B.14}$$

since $\nabla\cdot E=0$;

$$\nabla^2 E=\mu_o\epsilon_o\frac{\partial^2 E}{\partial^2 t} \tag{B.15}$$

and similarly,

$$\nabla\times(\nabla\times B)=\nabla(\nabla\cdot B)-\nabla^2 B \;=\; \nabla\times(\frac{\partial E}{\partial t}) \tag{B.16}$$

$$= -\frac{\partial}{\partial t}(\nabla\times E)=-\mu_o\epsilon_o\frac{\partial^2 B}{\partial^2 t} \tag{B.17}$$

since $\nabla\cdot B=0$;

$$\nabla^2 B=\mu_o\epsilon_o\frac{\partial^2 B}{\partial^2 t} \tag{B.18}$$

Which is the same as the wave equation,

$$\nabla^2 f=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{B.19}$$

the solution of which is,

$$f=f_o\sin(kx-\omega t) \tag{B.20}$$

Where the velocity v,

$$c=\frac{1}{\sqrt{\mu_0\epsilon_0}}, \tag{B.21}$$

is same as the speed of light $c=3.00\times10^8 m/s$, and

$$k \;=\; \frac{2pi}{\lambda} \tag{B.22}$$

$$\omega \;=\; 2\pi f \tag{B.23}$$

Appendix C

Alternate Derivations for Special Relativity

C.1 Building a suitable transformation: Einstein's way

Imagine a coordinate systems K with points (x, y, z, t) and K' with points (ξ, η, ζ, τ) moving with velocity v relative to K. At a given time, t, the origins of two coordinate systems are coincident, the axes are aligned and the velocity of K' is in the x, ξ direction. A light ray is emitted from the origin of k and reflected off of a point ξ' in k.

For every point (x, y, z, t) there is a transformation to (ξ, η, ζ, τ).

We make a note that x' is the distance from the origin O' to ξ' as seen in the frame K. (this is the same as the length of a moving rod measured in the stationary frame.) x' can be expressed as a function of time given by $x' = x - vt$

Because the origin in K' (O') and ξ' are stationary with respect to each other the condition of simultaneity holds for K',

$$\frac{1}{2}(\tau_0 + \tau_2) = \tau_1 \tag{C.1}$$

for a general transformation τ can be a function of (x', y, z, t), so we write the condition of simultaneity as,

$$\frac{1}{2}[\tau(0,0,0,t) + \tau(0,0,0,t + \frac{x'}{c-v} + \frac{x'}{c+v})] = \tau(x', 0, 0, t + \frac{x'}{c-v}). \tag{C.2}$$

If x' is very small we obtain (as differentiation),

$$\begin{aligned} \frac{1}{2}(\frac{1}{c-v} + \frac{1}{c+v})\frac{\partial \tau}{\partial t} &= \frac{\partial \tau}{\partial x'} + \frac{1}{c-v}\frac{\partial \tau}{\partial t} && \text{(C.3)} \\ \frac{\partial \tau}{\partial x'} + \frac{v}{c^2 - v^2}\frac{\partial \tau}{\partial t} &= 0 && \text{(C.4)} \end{aligned}$$

Homework 29
Derive: Eq (C.4) from Eq (C.3).
Solution:

$$\begin{aligned}
\frac{1}{2}(\frac{1}{c-v}+\frac{1}{c+v})+\frac{1}{c-v} &= \frac{1}{2}\Big(\frac{-(c+v)-(c-v)+2(c+v)}{c^2-v^2}\Big) && \text{(C.5)}\\
&= \frac{1}{2}\Big(\frac{-c-v-c+v+2c+2v)}{c^2-v^2}\Big) && \text{(C.6)}\\
&= \frac{1}{2}\frac{2v}{c^2-v^2}=\frac{v}{c^2-v^2} && \text{(C.7)}
\end{aligned}$$

Eq (C.4) implies a transformation of the form,

$$\tau = a(t-\frac{v}{c^2-v^2}x') \tag{C.8}$$

Where a is a yet unknown function of the velocity. Now make the substitution $x'=x-vt$ to get,

$$\begin{aligned}
\tau &= a(t-\frac{vx-tv^2}{c^2-v^2}) && \text{(C.9)}\\
&= a(\frac{c^2t-v^2t-vx+v^2t}{c^2-v^2}) = a(\frac{c^2(t-\frac{v}{c^2}x)}{c^2-v^2}) && \text{(C.10)}\\
\tau &= \frac{a}{1-\frac{v^2}{c^2}}(t-\frac{v}{c^2}x) && \text{(C.11)}
\end{aligned}$$

Similarly, in the x,ξ-direction.

$$\begin{aligned}
\xi &= c\tau && \text{(C.12)}\\
\xi &= ac(t-\frac{v}{c^2-v^2}x') && \text{(C.13)}
\end{aligned}$$

make the substitution that the time required to move the distance x' is,

$$t=\frac{x'}{c-v}, \tag{C.14}$$

to get,

$$\xi = ac(t - \frac{v}{c^2 - v^2}x'), \tag{C.15}$$

$$= ac(\frac{x'}{c-v} - \frac{v}{c^2 - v^2}x'), \tag{C.16}$$

$$= ac(\frac{x'(c+v)}{c^2 - v^2} - \frac{v}{c^2 - v^2}x'), \tag{C.17}$$

$$= ac(\frac{x'c + x'v - x'v}{c^2 - v^2}), \tag{C.18}$$

$$= ac(\frac{x'c}{c^2 - v^2}), \tag{C.19}$$

$$= \frac{ac^2}{c^2 - v^2}x', \tag{C.20}$$

Now substituting, $x' = x - vt$ we obtain,

$$\xi = \frac{ac^2}{c^2 - v^2}(x - vt), \tag{C.21}$$

Homework 30
Derive Eq. (C.20) from Eq. (C.15) and the facts that $t = \frac{x'}{c-v}$, and $x' = x - vt$.

Similarly in the directions perpendicular to the relative movement of the two reference frames. (y and z).

$$\eta = c\tau \tag{C.22}$$

For a pulse of light moving entirely in the η direction. Its vertical speed however in the reference frame K is given by, v in the x direction and c in the direction of the light ray (now appearing to travel at an angle to the x-direction), is from the Pythagorean theorem,

$$u = \sqrt{c^2 - v^2} \tag{C.23}$$

where the time it takes the light ray to travel a distance y is given by,

$$t = \frac{y}{\sqrt{c^2 - v^2}}. \tag{C.24}$$

Thus, with $x' = 0$ in the K' frame, we write η as,

$$\eta = c\tau = ac(t - \frac{v}{c^2 - v^2}x') \tag{C.25}$$

$$= ac(\frac{y}{\sqrt{c^2 - v^2}}) \tag{C.26}$$

Similarly, ζ may be written as,

$$\zeta = ac(\frac{z}{\sqrt{c^2 - v^2}}) \tag{C.27}$$

The equations,

$$\begin{aligned}
\tau &= \frac{a}{1 - \frac{v^2}{c^2}}(t - \frac{v}{c^2}x), && \text{(C.28)}\\
\xi &= \frac{ac^2}{c^2 - v^2}(x - vt), && \text{(C.29)}\\
\eta &= ac(\frac{y}{\sqrt{c^2 - v^2}}), && \text{(C.30)}\\
\zeta &= ac(\frac{z}{\sqrt{c^2 - v^2}}) && \text{(C.31)}
\end{aligned}$$

give the Lorentz transformations.

Using the transformations, $\phi(v) = \frac{a}{\sqrt{1 - \frac{v^2}{c^2}}}$, and $\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$. The Lorentz transforms may be written as,

$$\begin{aligned}
\tau &= \phi(v)\beta(t - \frac{v}{c^2}x), && \text{(C.32)}\\
\xi &= \phi(v)\beta(x - vt), && \text{(C.33)}\\
\eta &= \phi(v)y, && \text{(C.34)}\\
\zeta &= \phi(v)z. && \text{(C.35)}
\end{aligned}$$

Homework 31
Find the transformations $\phi(v)$, and β that allow, Eqs (C.28)-(C.31) to be written as Eqs (C.32)-(C.35).

Homework 32
If we have a wave given from a pulse of light starting at coincident origins in k and K, then the wave if emitted in all directions should be spherical in both coordinate systems. Show,

$$x^2 + y^2 + z^2 = c^2t^2 \tag{C.36}$$

transforms via the Lorentz transformations to,

$$\xi^2 + \eta^2 + \zeta^2 = c^2\tau^2 \tag{C.37}$$

C.1.1 The nature of $\phi(v)$

We need to examine $\phi(v)$ in more detail. given the transformation of,

$$\tau = \phi(v)\beta(t - \frac{v}{c^2}x), \tag{C.38}$$

Let us introduce a 3rd set of coordinates K'' moving with velocity $-v$ with respect to K'. (This will make it stationary with respect to K). Now, transforming τ to t' gives,

$$t' = \phi(-v)\beta(-v)(\tau + \frac{v}{c^2}\xi), \tag{C.39}$$

Where ϕ and β are functions of v, from the form of β we know that it is invariant under directional changes in v so $\beta(v) = \beta(-v)$. Now, we express t' in terms of t,

$$\begin{aligned}
t' &= \phi(-v)\beta(-v)(\tau + \frac{v}{c^2}\xi), && \text{(C.40)}\\
&= \phi(-v)\beta\Big(\phi(v)\beta(t - \frac{v}{c^2}x) + \frac{v}{c^2}\phi(v)\beta(x - vt)\Big), && \text{(C.41)}\\
&= \phi(-v)\phi(v)\beta^2\Big(t - \frac{v}{c^2}x + \frac{v}{c^2}(x - vt)\Big), && \text{(C.42)}\\
&= \phi(-v)\phi(v)\beta^2\Big(t - \frac{v}{c^2}x + \frac{v}{c^2}x - \frac{v^2}{c^2}t\Big), && \text{(C.43)}\\
&= \phi(-v)\phi(v)\beta^2\Big(t - \frac{v^2}{c^2}t\Big), && \text{(C.44)}\\
&= \phi(-v)\phi(v)\beta^2\Big(1 - \frac{v^2}{c^2}\Big)t, && \text{(C.45)}\\
&= \phi(-v)\phi(v)\beta^2\Big(\frac{1}{\beta^2}\Big)t, && \text{(C.46)}\\
&= \phi(-v)\phi(v)t, && \text{(C.47)}\\
& && \text{(C.48)}
\end{aligned}$$

Because k' is stationary with respect to K their clocks should be synchronized, and $t' = t$. similarly, the transform of position gives,

$$x' = \phi(-v)\phi(v)x, \tag{C.49}$$

And we conclude that,

$$\phi(-v)\phi(v) = 1 \tag{C.50}$$

Examining a rod moving in the x-direction, whos long axis is in the y-direction. In its stationary frame (k), the end points of the rod fall at, $(\xi_1 = 0, \eta_1 = 0, \zeta_1 = 0)$ and $(\xi_2 = 0, \eta_2 = l, \zeta_2 = 0)$. It follows from the transformation that its end points in the K frame should fall at, $(x_1 = vt, y_1 = 0, z_1 = 0)$ and $(x_2 = vt, y_2 = \frac{l}{\phi(v)}, z_2 = 0)$. Since the direction of the velocity (+ or -) depends solely upon the bias of the observer we would expect the length in the moving frame be the same regardless of the direction of the velocity,

$$\frac{l}{\phi(v)} = \frac{l}{\phi(-v)}, \tag{C.51}$$

which leads to the conclusion with, Eq (C.50) that,

$$\phi(v) = \phi(-v) = 1. \tag{C.52}$$

This allows us to write the Lorentz transformations as,

$$\begin{aligned}
\tau &= \beta(t - \frac{v}{c^2}x), && \text{(C.53)}\\
\xi &= \beta(x - vt), && \text{(C.54)}\\
\eta &= y, && \text{(C.55)}\\
\zeta &= z. && \text{(C.56)}
\end{aligned}$$

and,

$$\begin{aligned}
t &= \beta(\tau + \frac{v}{c^2}\xi), && \text{(C.57)}\\
x &= \beta(\xi + v\tau), && \text{(C.58)}\\
y &= \eta, && \text{(C.59)}\\
z &= \zeta. && \text{(C.60)}
\end{aligned}$$

Note: most textbooks use the notation $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ instead of what we are calling β, while the factor $\frac{v^2}{c^2}$ is given the name $\beta = \frac{v^2}{c^2}$.

C.2 The Lorentz transformation as a rotation in the complex plane

This derivation is taken from Pathria.

In order to derive the Lorentz transformations, we make some observations,

1. Cartesian w/ aligned axes
2. Relative motion is defined in the direction of one axis x, x' axes, the velocity of S' with respect to S is v.
3. Coincident at $t = 0$, $\tau = 0$,

$$x = y = z = t \tag{C.61}$$

$$\xi = \eta = \zeta = \tau \tag{C.62}$$

at $t = 0$ a light pulse is emitted from the origin and forms a spherical wave front.

$$x^2 + y^2 + z^2 - (ct)^2 = 0, \tag{C.63}$$
$$\xi^2 + \eta^2 + \zeta^2 - (c\tau)^2 = 0 \tag{C.64}$$

The law of inertia must be valid for both reference frames, the transformation between the two should be of a linear, homogeneous type.

$$\eta = \phi(v)y, \tag{C.65}$$

$\phi(v)$ can only be a function of velocity because S' and S remain parallel. now consider a reference frame S'' moving with velocity with respect to S', then,

$$y'' = \phi(v)y' = \phi(-v)\phi(v)y \tag{C.66}$$

then $\phi(-v)\phi(v)$=1, since $y'' = y$.

Since the motion is orthogonal to the direction of movement. Motion to the left should be indistinguishable from motion to the right.

$$\phi(v) = \phi(-v) \tag{C.67}$$

Therefore $phi(v) = \pm 1$ Since there is no inversion of axes, we conclude that $\phi(v) = 1$. This leads us to write,

$$y' = y \tag{C.68}$$
$$z' = z \tag{C.69}$$

this implies,

$$x^2 - (ct)^2 = \xi^2 - (c\tau)^2 \tag{C.70}$$

define

$$ict = x_4 \tag{C.71}$$
$$ic\tau = \xi_4, \tag{C.72}$$

now we can write Eq. (C.70) as,

$$x_1^2 + x_4^2 = \xi_1^2 + \xi_4^2 \tag{C.73}$$

Now apply x_1 and x_4 as coordinates in the complex plane,

$$r = x_1^2 + x_4^2 \tag{C.74}$$

where r is the distance from (0,0) to the point. Since $\xi_1^2 + \xi_4^2$ is in the same form.

$$r = \xi_1^2 + \xi_4^2 \tag{C.75}$$

The new axes may be obtained by a rotation in the plane,

$$\xi_1 = x_1 \cos\phi + x_4 \sin\phi \tag{C.76}$$
$$\xi_4 = -x_1 \sin\phi + x_4 \cos\phi \tag{C.77}$$

ϕ is the angle of rotation and is a function of v. An object at rest in S' must have a velocity v in S.

$$\frac{d\xi_1}{d\xi_4} = 0, \tag{C.78}$$
$$\frac{dx_1}{dx_4} = -ivc \tag{C.79}$$

ADD SOME ALGEBRA

$$\frac{dx_1}{dx_4} = \frac{\frac{dx_1}{dx_4}\cos\phi + \sin\phi}{-\frac{dx_1}{dx_4}\sin\phi + \cos\phi} \tag{C.80}$$

This leads to

$$\tan\phi = \frac{iv}{c} \tag{C.81}$$

Therefore we write,

$$\sin\phi = \frac{iv/c}{\sqrt{1-\frac{v^2}{c^2}}} \tag{C.82}$$
$$\cos\phi = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{C.83}$$

This means we can write,

$$\xi = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{C.84}$$
$$\tau = \frac{t - \frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{C.85}$$

C.3 Velocity Transform

We can write dx and dt in terms of the primed variables as,

$$dx = \beta(dx' + vdt') \tag{C.86}$$
$$dt = \beta(dt' + \frac{v}{c^2}dx') \tag{C.87}$$

Now we write u_x as,

$$\begin{aligned} u_x &= \frac{dx}{dt} && \text{(C.88)} \\ &= \frac{\beta(dx' + vdt')}{\beta(dt' + \frac{v}{c^2}dx')} && \text{(C.89)} \\ &= \frac{dt'(\frac{dx'}{dt'} + v)}{dt'(1 + \frac{v}{c^2}\frac{dx'}{dt'})} && \text{(C.90)} \\ &= \frac{u'_x + v}{1 + \frac{v}{c^2}u'_x} && \text{(C.91)} \end{aligned}$$

To transform a velocity perpendicular to the relative motion, (y-direction) $dy = dy'$,

$$\begin{aligned} u_y = \frac{dy}{dt} &= \frac{dy'}{\beta(dt' + \frac{v}{c^2}dx')} && \text{(C.92)} \\ &= \frac{\frac{dy'}{dt'}}{\beta(1 + \frac{v}{c^2}\frac{dy'}{dt'})} && \text{(C.93)} \\ &= \frac{u'_y}{\beta(1 + \frac{v}{c^2}u'_x)} && \text{(C.94)} \end{aligned}$$

We can sum up the transformations as,

$$\begin{aligned} u_x &= \frac{u'_x + v}{1 + \frac{v}{c^2}u'_x}, && \text{(C.95)} \\ u_y &= \frac{u'_y}{\beta(1 + \frac{v}{c^2}u'_x)}, && \text{(C.96)} \\ u_z &= \frac{u'_z}{\beta(1 + \frac{v}{c^2}u'_x)}, && \text{(C.97)} \\ u'_x &= \frac{u_x - v}{1 - \frac{v}{c^2}u_x}, && \text{(C.98)} \\ u'_y &= \frac{u_y}{\beta(1 - \frac{v}{c^2}u_x)}, && \text{(C.99)} \\ u'_z &= \frac{u_z}{\beta(1 - \frac{v}{c^2}u_x)}. && \text{(C.100)} \\ & && \text{(C.101)} \end{aligned}$$

C.3.1 Example: transforming the velocity of an object moving at the speed of light

Now, let us apply this to an object moving at c in frame S'.

$$\begin{aligned} u'_x &= c & \text{(C.102)} \\ u_x &= \frac{u'_x + v}{1 + \frac{v}{c^2}u'_x}, & \text{(C.103)} \\ &= \frac{c+v}{1+\frac{v}{c^2}c}, & \text{(C.104)} \\ &= \frac{c+v}{\frac{1}{c}(c+v)} = c & \text{(C.105)} \end{aligned}$$

C.3.2 Transformation of momentum and energy from one frame to another

We can see that the reletivistic momentum is written as function of its relative velocity in frame, S'.

$$p' = \frac{m_0 u'}{\sqrt{1-\frac{u'^2}{c^2}}} \tag{C.106}$$

while in frame S, we write the momentum as,

$$p = \frac{m_0 u}{\sqrt{1-\frac{u^2}{c^2}}} \tag{C.107}$$

We know the rules for converting velocites, but we need to be able to convert, $\sqrt{1-\frac{u'^2}{c^2}}$ to $\sqrt{1-\frac{u^2}{c^2}}$.

We begin with the velocity transformations,

$$u_x = \frac{u'_x + v}{1 + \frac{v}{c^2}u'_x}, \quad (C.108)$$

$$u_y = \frac{u'_y}{\beta(1 + \frac{v}{c^2}u'_x)}, \quad (C.109)$$

$$u_z = \frac{u'_z}{\beta(1 + \frac{v}{c^2}u'_x)}, \quad (C.110)$$

$$u'_x = \frac{u_x - v}{1 - \frac{v}{c^2}u_x}, \quad (C.111)$$

$$u'_y = \frac{u_y}{\beta(1 - \frac{v}{c^2}u_x)}, \quad (C.112)$$

$$u'_z = \frac{u_z}{\beta(1 - \frac{v}{c^2}u_x)}. \quad (C.113)$$

$$(C.114)$$

Then we can write u'^2 as,

$$u'^2 = u'^2_x + u'^2_y + u'^2_z = \frac{(u_x - v)^2}{(1 - \frac{v}{c^2}u_x)^2} + \frac{u_y^2}{\beta^2(1 - \frac{v}{c^2}u_x)^2} + \frac{u_z^2}{\beta^2(1 - \frac{v}{c^2}u_x)^2} \quad (C.115)$$

$$u'^2 = \frac{1}{\beta^2(1 - \frac{v}{c^2}u_x)^2}\big(\beta^2(u_x - v)^2 + u_y^2 + u_z^2\big), \quad (C.116)$$

$$\frac{u'^2}{c^2} = \frac{1}{\beta^2(1 - \frac{v}{c^2}u_x)^2}\big(\frac{\beta^2}{c^2}(u_x - v)^2 + \frac{u_y^2}{c^2} + \frac{u_z^2}{c^2}\big), \quad (C.117)$$

$$1 - \frac{u'^2}{c^2} = \frac{1}{\beta^2(1 - \frac{v}{c^2}u_x)^2}\big(\beta^2(1 - \frac{v}{c^2}u_x)^2 - \frac{\beta^2}{c^2}(u_x - v)^2 - \frac{u_y^2}{c^2} - \frac{u_z^2}{c^2}\big), \quad (C.118)$$

Let us look at the u_x term for a moment,

$$\beta^2(1 - \frac{v}{c^2}u_x)^2 - \frac{\beta^2}{c^2}(u_x - v)^2 = \beta^2\Big[1 - \frac{2vu_x}{c^2} + \frac{V^2u_x^2}{c^4} - \frac{u_x^2}{c^2} + \frac{2vu_x}{c^2} - \frac{v^2}{c^2}\Big], \quad (C.119)$$

$$= \beta^2\Big[1 - \frac{u_x^2}{c^2} - \frac{v^2}{c^2} + \frac{V^2u_x^2}{c^4}\Big], \quad (C.120)$$

$$= \beta^2\Big(1 - \frac{v^2}{c^2}\Big)\Big(1 - \frac{u_x^2}{c^2}\Big), \quad (C.121)$$

$$= 1 - \frac{u_x^2}{c^2} \quad (C.122)$$

Substituing this result back into Eq (C.118),

$$1-\frac{u'^2}{c^2} = \frac{1}{\beta^2(1-\frac{v}{c^2}u_x)^2}\Big(1-\frac{u_x^2}{c^2}-\frac{u_y^2}{c^2}-\frac{u_z^2}{c^2}\Big), \tag{C.123}$$

$$1-\frac{u'^2}{c^2} = \frac{1}{\beta^2(1-\frac{v}{c^2}u_x)^2}\Big(1-\frac{u^2}{c^2}\Big), \tag{C.124}$$

$$\frac{1}{\sqrt{1-\frac{u'^2}{c^2}}} = \beta(1-\frac{v}{c^2}u_x)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \tag{C.125}$$

now we define,

$$g(u) = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \tag{C.126}$$

We now write Eq (C.125) as,

$$g(u') = \beta\Big(g(u) - g(u)\frac{vu_x}{c^2}\Big) \tag{C.127}$$

$$g(u) = \beta\Big(g(u') + g(u')\frac{vu_x}{c^2}\Big) \tag{C.128}$$

and Eqs. (C.106) and (C.107) as,

$$p' = m_0u'g(u'), \tag{C.129}$$

$$p = m_0ug(u), \tag{C.130}$$

Now we have the math in place to complete the transformation of the momentum from the S' to the S frame. We will transform each carteasian coordinate seperately,

$$p_x = m_0u_xg(u), \tag{C.131}$$

$$p'_x = m_0u'_xg(u') \tag{C.132}$$

$$= m_0(\frac{u_x-v}{1-\frac{v}{c^2}u_x})\beta g(u)\Big(1-\frac{vu_x}{c^2}\Big), \tag{C.133}$$

$$= m_0(u_x-v)\beta g(u) = \beta(m_0u_xg(u) - \frac{v}{u}m_0ug(u)), \tag{C.134}$$

$$p'_x = \beta(p_x - \frac{V}{u}p). \tag{C.135}$$

Recall,

$$\frac{u}{c} = \frac{pc}{E} \tag{C.136}$$

$$u = \frac{pc^2}{E} \tag{C.137}$$

Now we finally write,

$$p'_x = \beta(p_x - \frac{V}{u}p), \tag{C.138}$$
$$= \beta(p_x - \frac{V}{c^2}E), \tag{C.139}$$

Transforming momentums in the $y-$direction (and z-direction)

$$p_y = m_0 u_y g(u), \tag{C.140}$$
$$p'_y = m_0 u'_y g(u') \tag{C.141}$$
$$p'_y = m_0 u_y g(u) \frac{\beta(1 - \frac{vu_x}{c^2})}{\beta(1 - \frac{vu_x}{c^2})}, \tag{C.142}$$
$$p'_y = m_0 u_y g(u) = p_y \tag{C.143}$$

Transforming the energy requires a bit more insight, $E = mc^2$, start with the transform,

$$g(u') = \beta\Big(g(u) - g(u)\frac{vu_x}{c^2}\Big) \tag{C.144}$$

multiply by m_0 to find the transform between relativistic masses,

$$g(u') = \beta\Big(g(u) - g(u)\frac{vu_x}{c^2}\Big), \tag{C.145}$$
$$m_0 g(u') = \beta\Big(m_0 g(u) - m_0 g(u)\frac{vu_x}{c^2}\Big), \tag{C.146}$$
$$m' = \beta\Big(m - \frac{v}{c^2}p_x\Big), \tag{C.147}$$
$$\frac{E'}{c^2} = \beta\Big(\frac{E}{c^2} - \frac{v}{c^2}p_x\Big). \tag{C.148}$$